

REPRESENTATION THEORETIC REALIZATION OF NON-SYMMETRIC MACDONALD POLYNOMIALS AT INFINITY

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ABSTRACT. We study the nonsymmetric Macdonald polynomials specialized at infinity from various points of view. First, we define a family of modules of the Iwahori algebra whose characters are equal to the nonsymmetric Macdonald polynomials specialized at infinity. Second, we show that these modules are isomorphic to the dual spaces of sections of certain sheaves on the semi-infinite Schubert varieties. Third, we prove that the global versions of these modules are homologically dual to the level one affine Demazure modules.

1. INTRODUCTION

Nonsymmetric Macdonald polynomials $E_\lambda(x, q, t)$ form a remarkable class of special functions (see [O, M3, Ch1, Ch2]). They depend on a weight of a simple Lie algebra \mathfrak{g} and variables $x = (x_1, \dots, x_n)$, q and t . Each $E_\lambda(x, q, t)$ is a polynomial in x -variables with coefficients being rational functions in q and t . The importance of the nonsymmetric Macdonald polynomials comes from numerous applications in combinatorics, algebraic geometry and representation theory. In particular, it has been shown in [S, I] that the characters of the affine level one Demazure modules for the corresponding affine Kac-Moody Lie algebra are equal to the $t = 0$ specializations $E_\lambda(x, q, 0)$.

It has been demonstrated recently that the $t = \infty$ specialization of the nonsymmetric Macdonald polynomials is very meaningful as well (see [CO1, CO2, OS, Kat, FeMa3, NS, NNS]). The study of the "opposite" $t = \infty$ specialization has lead to various discoveries of representation theoretic, combinatorial and geometric nature. However, all the representation theoretic descriptions of $E_\lambda(x, q, \infty)$ obtained so far are dealing only with the nonsymmetric Macdonald polynomials corresponding to the anti-dominant weight λ (recall that the Sanderson and the Ion theorems work for arbitrary λ). The goal of this paper is to fill this gap and to present the representation theoretic realization of $E_\lambda(x, q, \infty)$ for all weights.

Our starting point is a result from [Kat] stating that there exists a geometric realization of all the nonsymmetric Macdonald polynomials at $t = \infty$. More precisely, it has been proved that for any dominant weight λ and an element $w \in W$ there exists a sheaf $\mathcal{E}_w(\lambda)$ on the semi-infinite Schubert variety $\Omega(w)$ (see e.g. [BF1, BF2, Kat, KNS]) such that the character of the dual space of sections of $\mathcal{E}_w(\lambda)$ is equal (up to a simple factor) to the

nonsymmetric Macdonald polynomial $E_{-w\lambda}(x^{-1}, q^{-1}, \infty)$ (see Section 4 for more details). Moreover, this space is naturally endowed with the structure of a cyclic module over the Iwahori algebra. Our first main result is an explicit description of the corresponding module of the Iwahori. Namely, we put forward the following definition:

Definition 1.1. Let λ_- be an anti-dominant weight and let σ be an element of the Weyl group of \mathfrak{g} . The module $U_{\sigma(\lambda_-)}$ is the cyclic Iwahori module with cyclic vector $u_{\sigma(\lambda_-)}$ of \mathfrak{h} weight $\sigma(\lambda_-)$ subject to the relations:

$$\begin{aligned} \mathfrak{h} \otimes z\mathbb{C}[z]u_{\sigma(\lambda_-)} &= 0, \\ \widehat{\sigma}(f_{-\alpha} \otimes z)u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \\ (f_{\sigma(\alpha)} \otimes z)^{-\langle \lambda_-, \alpha^\vee \rangle + 1} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_-, \\ (e_{\sigma(\alpha)} \otimes 1)^{-\langle \lambda_-, \alpha^\vee \rangle} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_+, \end{aligned}$$

where the definition of the $\widehat{\sigma}$ -action is given in §2.2. The global version $\mathbb{U}_{\sigma(\lambda_-)}$ is defined by the same set of relations with the first line omitted. We prove the following theorem:

Theorem 1.2. *Let \mathfrak{g} be of type ADE. Then for an anti-dominant weight λ_- and $\sigma \in W$ one has*

$$E_{\sigma(\lambda_-)}(x, q^{-1}, \infty) = w_0 \text{ch } U_{w_0\sigma(\lambda_-)}.$$

We conjecture that this theorem is true for all types, but we do not have a general proof.

The above U -modules also give the spaces of sections of the sheaves $\mathcal{E}_w(\lambda)$ on a Schubert manifold $\mathfrak{Q}(w)$. More precisely, we prove the following theorem (holds for an arbitrary \mathfrak{g}).

Theorem 1.3. *For a dominant weight λ and $w \in W$ one has an isomorphism of the Iwahori modules*

$$H^0(\mathfrak{Q}(w), \mathcal{E}_w(\lambda))^* \simeq \mathbb{U}_{w(\lambda)}.$$

For an antidominant weight μ we consider a series $(q)_\mu^{-1} \in \mathbb{C}[[q]]$ (see section 2.2). In view of [Kat, Corollary 6.10], Theorem 1.3 implies

Corollary 1.4. *For an anti-dominant weight λ_- and $\sigma \in W$ one has*

$$(q)_{(\lambda_-)_\sigma}^{-1} \cdot E_{\sigma(\lambda_-)}(x, q^{-1}, \infty) = w_0 \text{ch } \mathbb{U}_{w_0\sigma(\lambda_-)},$$

where $(\lambda_-)_\sigma$ is defined by (3.1).

Our third theorem describes the categorical nature of the global U -modules. Let \mathfrak{B} be the category of the Iwahori modules (see section 5 for the precise definitions). Let D_μ be the level one affine Demazure module whose cyclic vector has weight μ . In particular, thanks to [S, I] the character of D_μ is given by the $t = 0$ specialization of the nonsymmetric Macdonald polynomial E_μ . We note that D_μ as well as U_λ are elements of \mathfrak{B} . We prove that the global U -modules are "dual" in the categorical sense to the Demazure modules. More precisely, the following theorem holds.

Theorem 1.5. *Assume that \mathfrak{g} is of type ADE_6E_7 . We have:*

$$\mathrm{Ext}_{\mathfrak{g}}^i(\mathbb{U}_{-\lambda}, D_{\mu}^*) \cong \begin{cases} \mathbb{C} & (i = 0, \lambda = \mu) \\ \{0\} & (\text{otherwise}). \end{cases}$$

We conjecture that the theorem holds for other types as well.

Our paper is organized as follows. In Section 2 we collect main definitions we use in the main body of the paper. In Section 3 we study the representation theory of the local and global U -modules. In Section 3.4 the link between the representation theoretical properties of the modules U_{μ} and the combinatorics of the nonsymmetric Macdonald polynomials is established; in particular, we prove Theorem 1.2. Section 4 contains the study of the geometry of the semi-infinite Schubert varieties; in particular, we prove Theorem 1.3. Finally, Section 5 is devoted to the study of the categorical properties of the modules $\mathbb{U}_{w\lambda}$ and to the proof of Theorem 1.5. A combinatorial consequence of Theorem 1.5 is discussed in the Appendix.

2. PRELIMINARIES

For a \mathbb{Z} -graded vector space $V = \bigoplus_{m \in \mathbb{Z}} V_m$, we set

$$\mathrm{gdim} V := \sum_{m \in \mathbb{Z}} q^m \dim V_m,$$

that is a priori a formal sum. We also define $V^* := \bigoplus_{m \in \mathbb{Z}} V_{-m}^*$, where its degree m -part is understood to be V_{-m}^* .

2.1. Finite dimensional objects. Let \mathfrak{g} be a simple Lie algebra of rank n with the Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Let $\Delta = \Delta_+ \sqcup \Delta_- \subset \mathfrak{h}^*$ be the set of roots and let Q be the root lattice spanned by Δ . We set $I := \{1, 2, \dots, n\}$. We denote by $\{\alpha_i\}_{i \in I}$ the set of simple roots and by $\{\omega_i\}_{i \in I}$ the set of fundamental weights. For a root $\alpha \in \Delta$, we denote by $\alpha^\vee \in \mathfrak{h}$ the corresponding coroot. For the standard pairing $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \rightarrow \mathbb{C}$ one has $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

For each $\alpha \in \Delta_+$, we denote by $e_\alpha \in \mathfrak{n}_+$ the corresponding Chevalley generator of \mathfrak{g} . Similarly, for $\alpha \in \Delta_-$, we denote by f_α the Chevalley generator of weight α in \mathfrak{n}_- . Let $P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$ be the weight lattice with the dominant cone $P_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$. We set $P_- := -P_+$. For $\lambda \in P_+$, we denote the corresponding irreducible finite-dimensional highest weight \mathfrak{g} -module by $V(\lambda)$. Let $v \in V(\lambda)$ be a non-zero highest weight vector. Then $\mathfrak{n}_+v = 0$ and the defining relations of $V(\lambda)$ as \mathfrak{n}_- module are of the form

$$f_{-\alpha}^{\langle \lambda, \alpha^\vee \rangle + 1} v = 0, \quad \alpha \in \Delta_+.$$

Let \mathfrak{g}^\vee be the simple Lie algebra defined by the dual Kac-Moody data of \mathfrak{g} .

Finally, we denote by W the finite Weyl group of \mathfrak{g} . For a root α , the corresponding reflection is denoted by $s_\alpha \in W$. For each $i \in I$, we set

$s_i := s_{\alpha_i}$. We sometimes identify s_α with s_{α^\vee} as the Weyl groups of \mathfrak{g} and \mathfrak{g}^\vee coincide.

2.2. Current algebras. Let $\mathfrak{g}[z] = \mathfrak{g} \otimes \mathbb{C}[z]$ be the current algebra. We have a grading on $\mathfrak{g}[z]$ by setting $\deg a \otimes z^m = m$ for each $a \in \mathfrak{g} \setminus \{0\}$ and $m \geq 0$. For $\lambda \in P_+$, we define the local Weyl module $W(\lambda)$ of $\mathfrak{g}[z]$ as the cyclic $\mathfrak{g}[z]$ -module with a cyclic vector w of \mathfrak{h} -weight λ subject to the relations $\mathfrak{h} \otimes z\mathbb{C}[z]w = 0$, $\mathfrak{n}_+ \otimes \mathbb{C}[z]w = 0$ and

$$(f_{-\alpha} \otimes 1)^{\langle \lambda, \alpha^\vee \rangle + 1} w = 0, \quad \alpha \in \Delta_+.$$

We define the global Weyl module $\mathbb{W}(\lambda)$ of $\mathfrak{g}[z]$ by omitting the condition $\mathfrak{h} \otimes z\mathbb{C}[z]w = 0$. The characters of the local and global Weyl modules differ by a simple factor. Namely, let us consider the subspace $A(\lambda)$ of weight λ vectors in $\mathbb{W}(\lambda)$. Being a quotient of $U(\mathfrak{h} \otimes \mathbb{C}[z])$ by a homogeneous ideal, the vector space $A(\lambda)$ carries a structure of a graded commutative algebra whose grading is induced by the grading of $\mathfrak{g}[z]$. We set $r_i = \langle \lambda, \alpha_i^\vee \rangle$ for each $i \in I$. Then, the following holds:

- The algebra $A(\lambda)$ is isomorphic to the polynomial algebra in variables $\{x_{i,a}\}_{i \in I, 1 \leq a \leq r_i}$ of degree one, symmetric in each group of variables $x_{i,1}, \dots, x_{i,r_i}$;
- The algebra $A(\lambda)$ acts freely on the global Weyl module $\mathbb{W}(\lambda)$. The action commutes with the action of $\mathfrak{g}[z]$;
- One has $\mathbb{W}(\lambda)/\mathfrak{m}\mathbb{W}(\lambda) \simeq W(\lambda)$, where \mathfrak{m} is the ideal of A_{λ_+} consisting of polynomials without a constant term.

Let $\mathfrak{J} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{g} \otimes z\mathbb{C}[z] \subset \mathfrak{g} \otimes \mathbb{C}[z]$ be the Iwahori subalgebra.

Remark 2.1. Note that the Cartan subalgebra \mathfrak{h} is included in \mathfrak{J} . In [FeMa3, FeMa4, FMO] the authors used the algebra $\mathfrak{n}^{af} = \mathfrak{n}^+ \oplus \mathfrak{g} \otimes z\mathbb{C}[z] \subset \mathfrak{g} \otimes \mathbb{C}[z]$ instead of \mathfrak{J} . The only difference is that the Cartan part \mathfrak{h} is missing in \mathfrak{n}^{af} , i.e. $\mathfrak{J} = \mathfrak{n}^{af} \oplus \mathfrak{h}$.

An \mathfrak{J} module M is called graded, if $M = \bigoplus_{j \in \mathbb{Z}} M_j$ such that each M_j is \mathfrak{h} semi-simple (i.e. each M_j is the sum of \mathfrak{h} weight spaces) and $(x \otimes z^i)M_j \subset M_{i+j}$ for all $x \in \mathfrak{g}$, $i \geq 0$. We define the character of M as the formal linear combination

$$\text{ch } M = \sum_{j \in \mathbb{Z}} q^j \text{ch } M_j,$$

there $\text{ch } M_j$ is the \mathfrak{h} -module character. In what follows we always consider the modules M whose \mathfrak{h} -weights belong to P . We say that $\text{ch } M$ is well-defined whenever we have $\text{ch } M_j \in \mathbb{Z}[P]$ for each $j \in \mathbb{Z}$. We set $x_i = e^{\omega_i}$. Then we have $\text{ch } M \in \mathbb{Z}[x_1^\pm, \dots, x_n^\pm][[q, q^{-1}]]$ when $\text{ch } M$ is well-defined. If M is cyclic with cyclic vector v , then we assume that $v \in M_0$ unless stated otherwise.

One concludes that

$$\text{ch } \mathbb{W}(\lambda) = (q)_\lambda^{-1} \cdot \text{ch } W(\lambda),$$

where we have

$$(q)_\lambda = \prod_{i=1}^n \prod_{j=1}^{r_i} (1 - q^j) = (\text{gdim } A(\lambda))^{-1} \in \mathbb{Z}[[q]].$$

Extending this, we set $(q)_{w\lambda} := (q)_\lambda$ for each $w \in W$.

For each $\mu \in P$, let us denote by $E_\mu(x, q, t) \in \mathbb{C}[P](q, t)$ the non-symmetric Macdonald polynomial in the sense of Cherednik [Ch1]. In particular, the character of $\mathbb{W}(\lambda)$ for $\lambda \in P_+$ is given by

$$\text{ch } \mathbb{W}(\lambda) = (q)_\lambda^{-1} \cdot E_{w_0\lambda}(x, q, 0),$$

(cf. [S, I, FL]).

Recall the W -action on the set $\{e_\alpha\}_{\alpha \in \Delta_+} \cup \{f_{-\alpha} \otimes z\}_{\alpha \in \Delta_+}$ following [FeMa3]: for an element $\sigma \in W$ and $\alpha \in \Delta_+$ we set

$$\widehat{\sigma}e_\alpha = \begin{cases} e_{\sigma(\alpha)}, & \sigma(\alpha) \in \Delta_+, \\ f_{\sigma(\alpha)} \otimes z, & \sigma(\alpha) \in \Delta_-, \end{cases} \quad \widehat{\sigma}(f_{-\alpha} \otimes z) = \begin{cases} e_{-\sigma(\alpha)}, & \sigma(\alpha) \in \Delta_-, \\ f_{-\sigma(\alpha)} \otimes z, & \sigma(\alpha) \in \Delta_+. \end{cases}$$

2.3. Affine algebras. The affine Weyl group W^a and the extended affine Weyl group W^e attached to \mathfrak{g}^\vee fits into the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & Q & \longrightarrow & W^a & \longrightarrow & W \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & P & \xrightarrow{t} & W^e & \longrightarrow & W \longrightarrow 1 \end{array}$$

where the action of W on Q and P are the standard ones. In particular, every element $w \in W^e$ can be uniquely written as $w = t(\text{wt}(w))\text{dir}(w)$, where $\text{wt}(w) \in P$ and $\text{dir}(w) \in W$. In what follows, we sometimes write t_μ for the image of $\mu \in P$ through the map t .

Let $\widehat{\mathfrak{g}}^\vee$ be the untwisted affine Kac-Moody algebra corresponding to \mathfrak{g}^\vee . The real roots of this dual affine Kac-Moody algebra are of the form $\beta = \bar{\beta} + \delta^\vee \deg \beta$, where $\bar{\beta} \in \Delta^\vee$ and δ^\vee is the primitive null root. We sometimes call the roots of $\widehat{\mathfrak{g}}^\vee$ the affine coroots. Let us denote by $\widetilde{\Delta}_+^a$ the set of real positive affine coroots, and denote by Δ_+^a the set of real positive affine roots. For an element $w \in W^e$, the length $\ell(w)$ is defined as

$$\ell(w) := \#\{\beta \in \widetilde{\Delta}_+^a \mid w(\beta) \notin \widetilde{\Delta}_+^a\}.$$

The set of length zero elements is denoted by Π . One has a semi-direct product decomposition $W^e = \Pi \rtimes W^a$.

It is standard that the positive level affine action of W^a on $P \otimes_{\mathbb{Z}} \mathbb{R}$ identifies $P \otimes_{\mathbb{Z}} \mathbb{R}$ with the (closure of the) union of the W^a -translations of the fundamental region called the fundamental alcove [Lus]. In the same fashion, we regard $\Pi \times (P \otimes_{\mathbb{Z}} \mathbb{R})$ as the (closure of the) union of the W^a -translations of the fundamental alcove. For each $\pi \in \Pi$, we refer to $\pi \times (P \otimes_{\mathbb{Z}} \mathbb{R})$ as a sheet. The set of alcoves contained in $\pi \times (P \otimes_{\mathbb{Z}} \mathbb{R})$ is in bijection with πW^a .

Let $u(\lambda)$ be the minimal length element in the coset $t_\lambda W$. We have $t_\lambda = u(\lambda)$ if and only if $\lambda \in P_-$ (see e.g. [M3, (2.4.5)]). Let $v(\lambda) \in W$ be the shortest element such that $v(\lambda)\lambda \in P_-$. For $\lambda_- = v(\lambda)\lambda$ one has

$$t_\lambda = u(\lambda)v(\lambda) \quad \text{and} \quad t_{\lambda_-} = v(\lambda)u(\lambda).$$

One also has $\ell(t_{\lambda_-}) = \ell(v(\lambda)) + \ell(u(\lambda))$.

2.4. Graded homomorphisms. By a graded abelian category \mathfrak{C} , we mean an abelian category \mathfrak{C} equipped with an autoequivalence $M \mapsto M \langle n \rangle$ for each $M \in \mathfrak{C}$ and $n \in \mathbb{Z}$ so that we have a functorial isomorphism

$$(M \langle n \rangle) \langle -n \rangle \cong M.$$

In this setting, we define

$$\text{hom}_{\mathfrak{C}}(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathfrak{C}}(M \langle n \rangle, N) \quad M, N \in \mathfrak{C}.$$

We regard $\text{hom}_{\mathfrak{C}}(M, N)$ as a \mathbb{Z} -graded vector space whose n -th graded piece is given by

$$\text{hom}_{\mathfrak{C}}(M, N)_n := \text{Hom}_{\mathfrak{C}}(M \langle n \rangle, N).$$

In case \mathfrak{C} has enough projectives, then we also define

$$\text{ext}_{\mathfrak{C}}^i(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Ext}_{\mathfrak{C}}^i(M \langle n \rangle, N).$$

They have the usual long exact sequences associated to short exact sequences (degreewise). For a graded abelian category \mathfrak{C} , we denote by $[\mathfrak{C}]$ its Grothendieck group. The group $[\mathfrak{C}]$ naturally admits a $\mathbb{Z}[q, q^{-1}]$ -module structure by identifying the action of $\langle 1 \rangle$ with q .

3. U-MODULES

3.1. Definitions. For an anti-dominant weight λ_- and an element $\sigma \in W$ we define the local module $U_{\sigma(\lambda_-)}$ and the global module $\mathbb{U}_{\sigma(\lambda_-)}$ as follows. $U_{\sigma(\lambda_-)}$ is the cyclic \mathfrak{J} module with cyclic vector $u_{\sigma(\lambda_-)}$ of \mathfrak{h} weight $\sigma(\lambda_-)$ subject to the relations:

$$\begin{aligned} \mathfrak{h} \otimes z\mathbb{C}[z]u_{\sigma(\lambda_-)} &= 0, \\ \widehat{\sigma}(f_{-\alpha} \otimes z)u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \\ (f_{\sigma(\alpha)} \otimes z)^{-\langle \lambda_-, \alpha^\vee \rangle + 1} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_-, \\ (e_{\sigma(\alpha)} \otimes 1)^{-\langle \lambda_-, \alpha^\vee \rangle} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_+. \end{aligned}$$

The definition of the global \mathfrak{J} -module $\mathbb{U}_{\sigma(\lambda_-)}$ differs from the definition of the local module by removing the first line relation.

Remark 3.1. Let \mathfrak{g} be of type ADE and let $D_{\sigma(\lambda_-)}$ be the level one affine Demazure module whose cyclic vector has weight $\sigma(\lambda_-)$. Then $D_{\sigma(\lambda_-)}$ is

the cyclic \mathfrak{J} -module with the set of relations as for $U_{\sigma(\lambda_-)}$ with the last two lines replaced with

$$\begin{aligned} (f_{\sigma(\alpha)} \otimes z)^{-\langle \lambda_-, \alpha^\vee \rangle} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_-, \\ (e_{\sigma(\alpha)} \otimes 1)^{-\langle \lambda_-, \alpha^\vee \rangle + 1} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_+. \end{aligned}$$

i.e. $+1$ got moved to the last line relation.

Remark 3.2. Let $W_{\sigma(\lambda_-)}$ be the generalized Weyl module ([FeMa3, FMO, No]). Then $W_{\sigma(\lambda_-)}$ is the cyclic \mathfrak{J} -module with the set of relations as for $U_{\sigma(\lambda_-)}$ with the last two lines replaced with

$$\begin{aligned} (f_{\sigma(\alpha)} \otimes z)^{-\langle \lambda_-, \alpha^\vee \rangle + 1} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_-, \\ (e_{\sigma(\alpha)} \otimes 1)^{-\langle \lambda_-, \alpha^\vee \rangle + 1} u_{\sigma(\lambda_-)} &= 0, \quad \alpha \in \Delta_+, \sigma\alpha \in \Delta_+. \end{aligned}$$

i.e. $+1$ is now present in both relations.

Corollary 3.3. *We have natural surjections of \mathfrak{J} -modules $W_{\sigma(\lambda_-)} \rightarrow D_{\sigma(\lambda_-)}$ and $W_{\sigma(\lambda_-)} \rightarrow U_{\sigma(\lambda_-)}$.*

Remark 3.4. Let $\sigma = e$ be the identity element. Then the surjection $W_{\sigma(\lambda_-)} \rightarrow D_{\sigma(\lambda_-)}$ is an isomorphism. Let $\sigma = w_0$ be the longest element. Then the surjection $W_{\sigma(\lambda_-)} \rightarrow U_{\sigma(\lambda_-)}$ is an isomorphism.

Theorem 3.5 (Feigin-Makedonskyi [FeMa3, FMO, Kat]). *For $\lambda \in P_+$, the character of \mathbb{W}_λ is given by*

$$\text{ch } \mathbb{W}_\lambda = (q)_\lambda^{-1} \cdot w_0 E_{w_0 \lambda}(x, q^{-1}, \infty).$$

3.2. From local to global. Now assume that λ_-, σ satisfy $\langle \lambda_-, \alpha_i^\vee \rangle < 0$ provided $\sigma\alpha_i \in \Delta_+$ for $i \in I$. We define

$$(3.1) \quad (\lambda_-)_\sigma = \lambda_- + \sum_{j: \sigma\alpha_j > 0} \omega_j \in P_-.$$

Remark 3.6. In [Kat] for a dominant weight λ_+ and $w \in W$ the weight $(\lambda_+)_w$ is defined as $\lambda_+ - \sum_{j: w\alpha_j < 0} \omega_j$. This is the effect of the notation change from [Kat] to [FeMa3, FMO]: $w \rightarrow \sigma w_0$, $\lambda_+ \rightarrow w_0 \lambda_-$. Note that, first, $\sigma \lambda_- = w \lambda_+$ and, second, for an anti-dominant λ_- and $\sigma = w w_0 \in W$ one has

$$w_0(\lambda_- + \sum_{j: (w w_0)\alpha_j > 0} \omega_j) = w_0 \lambda_- + \sum_{w(w_0 \alpha_j) > 0} w_0 \omega_j = \lambda_+ - \sum_{w \alpha_{w_0 j} < 0} \omega_{w_0 j},$$

where we define the number $w_0 j$ by $\omega_{w_0 j} = -w_0 \omega_j$. So we conclude that $w_0(\lambda_-)_\sigma = (\lambda_+)_w$.

Lemma 3.7. *There exists surjective homomorphisms of \mathfrak{n}^{af} -modules $U_{\sigma(\lambda_-)} \rightarrow W_{\sigma((\lambda_-)_\sigma)}$ and (global version) $\mathbb{U}_{\sigma(\lambda_-)} \rightarrow \mathbb{W}_{\sigma((\lambda_-)_\sigma)}$.*

Proof. The proofs for local and global modules are the same, so we only do the global case. We have to show that all the defining relations of $\mathbb{U}_{\sigma(\lambda_-)}$ hold in $\mathbb{W}_{\sigma((\lambda_-)_\sigma)}$. So let $\alpha \in \Delta_+ \cap \sigma^{-1}\Delta_+$. Then we need to show that

$$(3.2) \quad (e_{\sigma(\alpha)} \otimes 1)^{-\langle \lambda_-, \alpha^\vee \rangle} w_{\sigma((\lambda_-)_\sigma)} = 0$$

in $\mathbb{W}_{\sigma((\lambda_-)_\sigma)}$. By definition

$$(e_{\sigma(\alpha)} \otimes 1)^{-\langle (\lambda_-)_\sigma, \alpha^\vee \rangle + 1} w_{\sigma((\lambda_-)_\sigma)} = 0$$

in $\mathbb{W}_{\sigma((\lambda_-)_\sigma)}$.

We assumed that σ satisfies $\langle \lambda_-, \alpha_i^\vee \rangle < 0$ provided $\sigma\alpha_i \in \Delta_+$ for $i \in I$ in the beginning of this subsection. Since $\sigma\alpha \in \Delta_+$, we have

$$-\langle (\lambda_-)_\sigma, \alpha^\vee \rangle + 1 \leq -\langle \lambda_-, \alpha^\vee \rangle$$

and (3.2) holds.

It remains to show that for $\alpha \in \Delta_+ \cap \sigma^{-1}\Delta_-$

$$(f_{\sigma(\alpha)} \otimes z)^{-\langle \lambda_-, \alpha^\vee \rangle + 1} w_{\sigma(\lambda_-)} = 0$$

in $\mathbb{W}_{\sigma((\lambda_-)_\sigma)}$. This follows from the obvious inequality $-\langle (\lambda_-)_\sigma, \alpha^\vee \rangle \leq -\langle \lambda_-, \alpha^\vee \rangle$ for any $\alpha \in \Delta_+$. \square

It is shown in [FMO] that the algebra of the weight $\sigma(\lambda_-)$ vectors of $\mathbb{W}_{\sigma(\lambda_-)}$ is isomorphic to $A(w_0\lambda_-)$ described in section 2 (in particular, independent of σ). In addition, it acts freely on $\mathbb{W}_{\sigma(\lambda_-)}$ and the quotient with respect to the no free term part is isomorphic to the local generalized Weyl module $W_{\sigma(\lambda_-)}$. One has $\text{ch } \mathbb{W}_{\sigma(\lambda_-)} = (q)_{\lambda_-}^{-1} \cdot \text{ch } W_{\sigma(\lambda_-)}$.

Proposition 3.8. *The character of the weight $\sigma(\lambda_-)$ vectors in $\mathbb{U}_{\sigma(\lambda_-)}$ is equal to $(q)_{(\lambda_-)_\sigma}^{-1}$.*

Proof. Lemma 3.7 gives the estimate: the character of the weight $\sigma(\lambda_-)$ vectors in $\mathbb{U}_{\sigma(\lambda_-)}$ is greater than or equal to the character of the weight $\sigma((\lambda_-)_\sigma)$ vectors in $\mathbb{W}_{\sigma((\lambda_-)_\sigma)}$. However, the second character is equal to $(q)_{(\lambda_-)_\sigma}^{-1}$.

To get the upper bound for the weight $\sigma(\lambda_-)$ vectors in $\mathbb{U}_{\sigma(\lambda_-)}$ we note that $A_{(\lambda_-)_\sigma}$ surjects onto the space of weight $\sigma(\lambda_-)$ vectors in $\mathbb{U}_{\sigma(\lambda_-)}$. In fact, for each α_i we have the relations in $\mathbb{U}_{\sigma(\lambda_-)}$

$$(e_{\sigma(\alpha_i)} \otimes 1)^{-\langle \lambda_-, \alpha_i^\vee \rangle} u_{\sigma(\lambda_-)} = (f_{-\sigma(\alpha_i)} \otimes z) u_{\sigma(\lambda_-)} = 0 \text{ if } \sigma(\alpha_i) \in \Delta_+,$$

$$(f_{\sigma(\alpha_i)} \otimes z)^{-\langle \lambda_-, \alpha_i^\vee \rangle + 1} u_{\sigma(\lambda_-)} = (e_{-\sigma(\alpha_i)} \otimes 1) u_{\sigma(\lambda_-)} = 0 \text{ if } \sigma(\alpha_i) \in \Delta_-,$$

which imply that the highest weight algebra of $\mathbb{U}_{\sigma(\lambda_-)}$ is a quotient of $A_{(\lambda_-)_\sigma}$. Now it suffices to note that $\text{ch } A_{(\lambda_-)_\sigma} = (q)_{(\lambda_-)_\sigma}^{-1}$. \square

Corollary 3.9. $\text{ch } \mathbb{U}_{\sigma(\lambda_-)} \leq (q)_{(\lambda_-)_\sigma}^{-1} \cdot \text{ch } U_{\sigma(\lambda_-)}$. *In case the equality holds, the $A_{(\lambda_-)_\sigma}$ -action on $\mathbb{U}_{\sigma(\lambda_-)}$ is free.*

Proof. We have the action of the algebra $A_{(\lambda_-)_\sigma}$ on the module $\mathbb{U}_{\sigma(\lambda_-)}$ (the proof repeats the proof of the analogous statements for classical global Weyl modules from [CFK] and for the generalized Weyl modules from [FeMa3]) making $\mathbb{U}_{\sigma(\lambda_-)}$ an $(U(\mathfrak{J}), A_{(\lambda_-)_\sigma})$ -bimodule. Let \mathfrak{m} be the ideal of $A_{(\lambda_-)_\sigma}$ consisting of polynomials without free term. Then we have:

$$\mathbb{U}_{\sigma(\lambda_-)} / \mathbb{U}_{\sigma(\lambda_-)} \cdot \mathfrak{m} \simeq U_{\sigma(\lambda_-)}.$$

Therefore we have:

$$\text{ch } \mathbb{U}_{\sigma(\lambda_-)} \leq \text{ch } U_{\sigma(\lambda_-)} \cdot \text{ch } A_{(\lambda_-)_\sigma}.$$

This completes the proof. Note that by the graded version of the Nakayama lemma the equality means that the action of $A_{(\lambda_-)_\sigma}$ is free. \square

3.3. The decomposition procedure. Let $\lambda_-, \mu \in P_-$ be such that $\lambda_- - \mu \in P_-$. We fix a reduced decomposition

$$(3.3) \quad t_\mu = \pi s_{j_1} \dots s_{j_l}, \quad \pi \in \Pi, \quad l = \ell(t_\mu)$$

in the extended affine Weyl group and consider the affine coroots β_1, \dots, β_l defined by

$$\beta_l(\mu) = \alpha_{j_l}^\vee, \beta_{l-1}(\mu) = s_{j_l} \alpha_{j_{l-1}}^\vee, \dots, \beta_1(\mu) = s_{j_l} \dots s_{j_2} \alpha_{j_1}^\vee$$

(see [OS]). In what follows we omit μ in the notation $\beta_j(\mu)$ if no confusion is possible. Recall the decomposition $\beta_j = \bar{\beta}_j + \delta^\vee \deg \beta_j$, where $\bar{\beta}_j \in \Delta^\vee$. We note that $\bar{\beta}_j$ is always a negative coroot and $\deg \beta_j > 0$. For a positive root α and a number $m = 1, \dots, l$ we define

$$(3.4) \quad l_{\alpha, m} = -\langle \lambda_-, \alpha^\vee \rangle - |\{j : \bar{\beta}_j = -\alpha^\vee, 1 \leq j \leq m\}|.$$

Then the generalized Weyl module with characteristics $W_{\sigma(\lambda_-)}(m)$ is the \mathfrak{n}^{af} module which is the quotient of $W_{\sigma(\lambda_-)}$ by the ideal generated by the elements

$$(3.5) \quad e_{\hat{\sigma}(\alpha)}^{l_{\alpha, m}+1}, \quad \alpha \in \Delta_+$$

(see [FeMa3, FMO]).

Remark 3.10. In order to make $W_{\sigma(\lambda_-)}(m)$ into an \mathfrak{J} -module, one has to specify the weight of the cyclic vector. If the opposite is not stated explicitly, we assume that the weight of the cyclic vector of $W_{\sigma(\lambda_-)}(m)$ is equal to $\sigma(\lambda_-)$, so that the natural surjection map $W_{\sigma(\lambda_-)} \rightarrow W_{\sigma(\lambda_-)}(m)$ is the \mathfrak{J} -modules homomorphism.

Lemma 3.11. *One has the natural chain of surjections of \mathfrak{n}^{af} -modules*

$$W_{\sigma(\lambda_-)} = W_{\sigma(\lambda_-)}(0) \rightarrow W_{\sigma(\lambda_-)}(1) \rightarrow \dots \rightarrow W_{\sigma(\lambda_-)}(l) = W_{\sigma(\lambda_- - \mu)}.$$

Proof. We note that for any positive root α the number of i ($1 \leq i \leq \ell(t_\mu)$) such that $\bar{\beta}_i = -\alpha^\vee$ is equal to $-\langle \mu, \alpha^\vee \rangle$. Now our lemma follows from (3.4) and the definition of the generalized Weyl modules with characteristics (see relations (3.5)). \square

In what follows we assume that \mathfrak{g} is simply-laced. We conjecture that all the statements below are true in general, but so far our proofs work in types A, D, E only.

Recall that the Bruhat graph BG of W (see e.g. [BB]) is the graph whose set of vertices is identified with W and we have an arrow $w \rightarrow ws_\alpha$ for $w \in W$ and $\alpha \in \Delta_+$ if and only if $\ell(ws_\alpha) = \ell(w) + 1$. The quantum Bruhat graph QBG of W (see e.g. [BFP, LNSSS1]) is an enhancement of BG obtained by adding a “quantum” arrow $w \rightarrow ws_\alpha$ for each $w \in W$ and $\alpha \in \Delta_+$ so that

$$\ell(ws_\alpha) = \ell(w) - \sum_{\gamma \in \Delta_+} \langle \gamma, \alpha^\vee \rangle + 1.$$

Lemma 3.12. *Assume there exists an arrow $\sigma \rightarrow \sigma s_{\bar{\beta}_m}$ in QBG. Then the kernel of the surjection $W_{\sigma(\lambda_-)}(m-1) \rightarrow W_{\sigma(\lambda_-)}(m)$ is a quotient of $W_{\sigma s_{\bar{\beta}_m}(\lambda_-)}(m-1)$. If the arrow does not exist, then the surjection $W_{\sigma(\lambda_-)}(m-1) \rightarrow W_{\sigma(\lambda_-)}(m)$ has no kernel.*

Proof. The case $\mu = -\omega_i$ for some i is proved in [FeMa3, Theorem 2.18]. The argument in [FeMa3] is as follows: we consider an arbitrary rank two root subsystems whose highest root is $-\bar{\beta}_m$ and check the consistency of the defining equations as modules over the corresponding (current algebra of) rank two Lie subalgebras.

The only statement one needs to modify in the proof in [FeMa3] in order to reduce the problem to the rank two case for arbitrary μ is:

$$(3.6) \quad |k \leq m : -\bar{\beta}_k = \gamma^\vee| = |k \leq m : -\bar{\beta}_k = \tau^\vee| + |k \leq m : -\bar{\beta}_k = \eta^\vee|,$$

where $0 \leq m \leq l = \ell(t_\mu)$, $\gamma^\vee = -\bar{\beta}_m$ and $\gamma = \tau + \eta$ for some $\tau, \eta \in \Delta$. The proof of (3.6) is in [FeMa3, Corollary 1.18 ii)] when $\mu = -\omega_i$.

We prove (3.6) for general μ in the ADE case. First, let \mathfrak{g} be of type A_2 . We note (see [OS], section 3.4) that the roots $-\bar{\beta}_1, \dots, -\bar{\beta}_l$ are exactly the labels of the walls crossed by the shortest path from the alcove corresponding to t_μ to the alcove π (the path belongs to the sheet corresponding to the length zero element π , see (3.3)). It is easy to see that the number of walls of type $\alpha_1 + \alpha_2$ crossed by such a path is equal to the number of crossed walls labeled by simple roots. Now assume that \mathfrak{g} is arbitrary simply-laced algebra. Given a decomposition $\gamma = \tau + \eta$ we restrict to the subsystem spanned by τ and η and forget all other walls. Then we use the type A_2 argument above.

After (3.6) is proved, the proof of the lemma goes along the same lines as the proof of Theorem 2.18 of [FeMa3]. Namely, one reduces the lemma to the rank two case. In types ADE this means that we restrict to a subalgebra of type A_2 or $A_1 \oplus A_1$. These types were worked out in [FeMa3], Section 3 for $\mu = -\omega_i$. Let us show that the result for $\mu = -\omega_i$ implies the general μ case (we do the A_2 case, the $A_1 \oplus A_1$ is even simpler). We use the following observation: for any antidominant μ any reduced expression for t_μ can be obtained as a concatenation of reduced expressions of $t_{-\omega_i}$. This implies

that the sequence of coroots $\{\beta_k(\mu)\}_k$ is obtained as concatenation of the sequences $\{\beta_k(-\omega_i)\}_k$, $i = 1, 2$. Hence, our Lemma holds for general μ , since all the generalized Weyl modules with characteristics showing up in the statement of Lemma 3.12 already appear for μ being negated fundamental weight. \square

Remark 3.13. Unfortunately, we are not able to prove Lemma 3.12 for non simply-laced algebras. The reason is twofold. First, we are not able to prove (3.6) in general. Second, we are not able to prove Lemma 3.12 in types C_2 and G_2 due to the fact that not all reduced decompositions of t_μ can be obtained as concatenation of reduced expressions of $t_{-\omega_i}$. Hence the generalized Weyl modules with characteristics showing up in Lemma 3.12 may a priori be different from the modules one gets for $\mu = -\omega_i$. We note that the second issue looks more subtle than the first one.

Lemma 3.14. *The kernel of the surjection $W_{\sigma(\lambda_-)}(m-1) \rightarrow W_{\sigma(\lambda_-)}(m)$ if non-trivial is isomorphic to $W_{\sigma s_{\bar{\beta}_m}(\lambda_-)}(m-1)$.*

Proof. The proof repeats that of [FeMa3, Theorem 2.21]. \square

Theorem 3.15. *Let μ be an anti-dominant weight with a reduced decomposition (3.3) and $\lambda_-, \lambda_- - \mu \in P_-$. For each $0 \leq m \leq l = \ell(t_\mu)$, the generalized Weyl module with characteristics $W_{\sigma(\lambda_-)}(m)$ can be filtered in such a way that:*

- *each subquotient is a generalized Weyl module of the form $W_{\tau(\lambda_- - \mu)}$ for some Weyl group element τ ;*
- *the number of subquotients is equal to the number of directed paths in the quantum Bruhat graph starting at σ and with labels of the form $\bar{\beta}_{j_1}, \dots, \bar{\beta}_{j_k}$, $m \leq j_1 < \dots < j_k \leq l$.*

Proof. The claim is a consequence of Lemma 3.12 and Lemma 3.14. \square

We now recall the notion of an alcove path (see [RY, OS]). Assume that we are given an element $z_0 \in W^e$ and a sequence of affine coroots β_1, \dots, β_l . Then a path p_J corresponding to a set $J = \{1 \leq j_1 < \dots < j_r \leq l\}$ is defined as $p_J = (z_0, z_1, \dots, z_r)$, where $z_{k+1} = z_k s_{\beta_{j_{k+1}}}$. The last element z_r is called the end of p ; we write $\text{end}(p) = z_r$. We say that p is a quantum alcove path if the projection $\text{dir} : W^e \rightarrow W$ induces the following path in the quantum Bruhat graph:

$$\text{dir}(z_0) \xrightarrow{\bar{\beta}_{j_1}} \text{dir}(z_1) \xrightarrow{\bar{\beta}_{j_2}} \dots \xrightarrow{\bar{\beta}_{j_r}} \text{dir}(z_r).$$

For an alcove path p we denote by $qwt^*(p)$ the sum of all β_{j_k} such that the edge $\text{dir}(z_{k-1}) \xrightarrow{\bar{\beta}_{j_k}} \text{dir}(z_k)$ is quantum.

Now let us take $\mu = \lambda_-$, fix $m = 0, \dots, \ell(t_\mu) - 1$ and the following sequence of affine coroots: $\beta_{m+1}(\lambda_-), \dots, \beta_l(\lambda_-)$, $l = \ell(t_{\lambda_-})$. Let $QB_{\sigma, \lambda_-}(m)$ be the set of quantum alcove paths with $z_0 = t_{\sigma(\lambda_-)}\sigma$ and the sequence of β 's given by $\beta_{m+1}(\lambda_-), \dots, \beta_l(\lambda_-)$. Then Lemma 3.12 and Lemma 3.14 provide a

combinatorial model for the generalized Weyl modules with characteristics $W_{\sigma(\lambda_-)}(m)$.

Corollary 3.16. *One has*

$$\text{ch} W_{\sigma(\lambda_-)}(m) = \sum_{p \in QB_{\sigma, \lambda_-}(m)} x^{\text{wt}(\text{end}(p))} q^{\deg(qwt^*(p))}.$$

Proof. Since $\mu = \lambda_-$ the subquotients in Theorem 3.15 are all one-dimensional. In addition, they are parametrized by the elements $p \in QB_{\sigma, \lambda_-}(m)$. So it suffices to show that the character of the one-dimensional subquotient labeled by p is given by $x^{\text{wt}(\text{end}(p))} q^{\deg(qwt^*(p))}$. The proof is contained in the proof of Corollary 2.9 of [FeMa3]. \square

Now let us consider the decomposition procedure for the global generalized Weyl modules. Recall that $\mathbb{W}_{\sigma(\lambda_-)}(m)$ is the \mathfrak{n}^{af} module which is the quotient of $\mathbb{W}_{\sigma(\lambda_-)}$ by the \mathfrak{n}^{af} submodules generated by the vectors $e_{\hat{\sigma}(\alpha)}^{l_{\alpha, m}+1} v$ for all $\alpha \in \Delta_+$, where v is the cyclic vector of $\mathbb{W}_{\sigma(\lambda_-)}$. The \mathfrak{n}^{af} -module structure can be extended to the \mathfrak{J} -module structure by fixing (in an arbitrary way) the weight of the cyclic vector. We assume that the weight of the cyclic vector of $\mathbb{W}_{\sigma(\lambda_-)}(m)$ is equal to $\sigma(\lambda_-)$ unless stated otherwise.

Recall that the character of a graded \mathfrak{J} -module M is denoted by $\text{ch } M = \sum_{\mu \in P} x^\mu c_\mu(q)$ for some series $c_\mu(q)$.

The following is a slight generalization of [FMO] from the case $\mu = -\omega_i$.

Theorem 3.17. *Let $\lambda_-, \mu, \lambda_- - \mu \in \Lambda_-$ and let $\beta_j = \beta_j(\mu)$. Then the following holds:*

- (i) *If there is no edge $\sigma \rightarrow \sigma s_{\bar{\beta}_m}$ in QBG, then the surjective map $\mathbb{W}_{\sigma(\lambda_-)}(m-1) \rightarrow \mathbb{W}_{\sigma(\lambda_-)}(m)$ is an isomorphism. If the edge does exist, then the kernel of this map is isomorphic to the generalized global Weyl module with characteristics $\mathbb{W}_{\sigma s_{\bar{\beta}_m}(\lambda_-)}(m-1)$.*

- (ii) *One has*

$$\text{ch } \mathbb{W}_{\sigma(\lambda_-)}(m) = \frac{\text{ch } W_{\sigma(\lambda_-)}(m)}{(q)_{\lambda_- + \omega(m)}},$$

where $\omega(m)$ is the sum of fundamental weights ω_{j_i} , where the sum is taken over all $1 \leq i \leq m$ such that $-\bar{\beta}_i = \alpha_{j_i}^\vee$ is simple.

Proof. The proof of the first claim of i) is analogous to that of [FeMa3, Theorem 2.18 i)]. In the same way as in Lemma 3.12 we prove that the kernel of the map $\mathbb{W}_{\sigma(\lambda_-)}(m-1) \rightarrow \mathbb{W}_{\sigma(\lambda_-)}(m)$ is a quotient of the module $\mathbb{W}_{\sigma s_{\bar{\beta}_m}(\lambda_-)}(m-1)$.

To prove part (ii) we use the same inductive argument as in the proof [FMO, Theorem 3.16] (that ultimately relies on the counting paths in the quantum Bruhat graph in [FeMa3, Theorem 2.21] through [FMO, Lemma 3.12]). The only modification needed is [FMO, Lemma 3.13]. Namely, the

crucial point in this Lemma is to figure out if a coroot $-\bar{\beta}_\bullet$ is simple. In particular, the proof of [FMO, Theorem 3.16] implies that if

$$\text{ch } \mathbb{W}_{\sigma(\lambda_-)}(j) = \frac{\text{ch } W_{\sigma(\lambda_-)}(j)}{(q)_\mu}$$

for some anti-dominant weight μ , then

$$\text{ch } \mathbb{W}_{\sigma(\lambda_-)}(j+1) = \begin{cases} \frac{\text{ch } W_{\sigma(\lambda_-)}(j+1)}{(q)_\mu}, & \text{if } -\bar{\beta}_{j+1} \text{ is not simple,} \\ \frac{\text{ch } W_{\sigma(\lambda_-)}(j+1)}{(q)_{\mu+\omega_i}}, & \text{if } -\bar{\beta}_{j+1} = \alpha_i^\vee. \end{cases}$$

If μ is equal to a negated fundamental weight, then $-\bar{\beta}_{j+1}$ is simple if and only if $j = 0$. For general μ there might be several simple roots among $\{-\bar{\beta}_j\}_{j=1}^m$. \square

3.4. Nonsymmetric Macdonald polynomials at infinity. The goal of this subsection is to identify the modules $U_{\sigma(\lambda_-)}$ with generalized Weyl modules with characteristics. The combinatorial model (Theorem 3.15) for the generalized Weyl modules with characteristics will allow to prove the equality between the character of $U_{\sigma(\lambda_-)}$ and the $t = \infty$ specialization of the corresponding nonsymmetric Macdonald polynomial.

In what follows we assume that σ is the maximal length element in the class $\sigma \cdot \text{stab}(\lambda_-) \subset W$ (recall that the modules $W_{\sigma(\lambda_-)}$ depend only on $\sigma(\lambda_-)$, but not on σ and λ_- separately). Let $\lambda'_- = w_0\sigma(\lambda_-)$, then $v(\lambda'_-) = \sigma^{-1}w_0$ is the shortest element such that $v(\lambda'_-)\lambda'_- = \lambda_-$. Moreover, the length formula [M3, (2.4.5)] implies that $t_{\lambda_-} = v(\lambda'_-)u(\lambda'_-)$ refines to a reduced expression, where $u(\lambda'_-)$ is the shortest element in the coset $t_{\lambda'_-}W$ (see section 2.3).

Let

$$v(\lambda'_-) = s_{i_1} \dots s_{i_r}, \quad u(\lambda'_-) = \pi s_{i_{r+1}} \dots s_{i_M}.$$

Then we obtain the reduced decomposition

$$(3.7) \quad t_{\lambda_-} = \pi s_{\pi^{-1}i_1} \dots s_{\pi^{-1}i_r} s_{i_{r+1}} \dots s_{i_M}.$$

Recall (see [OS]) that for $u \in W^e$ one denotes by $\mathcal{QB}(id; u)$ the set of quantum alcove paths with $z_0 = u$ and with β 's coming from a fixed reduced decomposition of u . Also one denotes by $\overleftarrow{\mathcal{QB}}(id; u)$ the set of alcove paths which project to some path in the reversed quantum Bruhat graph.

Theorem 3.18. *For $\beta_j = \beta_j(t_{\lambda_-})$ defined by the reduced decomposition (3.7) one has $U_{\sigma(\lambda_-)} \simeq W_{\sigma(\lambda_-)}(\ell(w_0) - \ell(\sigma))$. In addition, $\text{ch } W_{\sigma(\lambda_-)}(\ell(w_0) - \ell(\sigma)) = w_0 E_{w_0\sigma\lambda_-}(x, q^{-1}, \infty)$.*

Proof. We have to prove that the defining relations (3.5) of $W_{\sigma(\lambda_-)}(\ell(w_0) - \ell(\sigma))$ coincide with the defining relations of $U_{\sigma(\lambda_-)}$. It suffices to show that $\{-\bar{\beta}_1^\vee, \dots, -\bar{\beta}_r^\vee\} = \Delta_+ \cap \sigma^{-1}\Delta_+$ (we note that $r = \ell(v(\lambda'_-)) = \ell(\sigma^{-1}w_0) =$

$\ell(w_0) - \ell(\sigma)$, which is the cardinality of the set $\Delta_+ \cap \sigma^{-1}\Delta_+$. By definition, for $k = 1, \dots, r$ we have

$$(3.8) \quad \beta_k = s_{i_M} \dots s_{i_{r+1}} s_{\pi^{-1}i_r} \dots s_{\pi^{-1}i_{k+1}} \alpha_{\pi^{-1}i_k}^\vee$$

$$(3.9) \quad = t_{-\lambda_-} \pi s_{\pi^{-1}i_1} \dots s_{\pi^{-1}i_{k-1}} s_{\pi^{-1}i_k} \alpha_{\pi^{-1}i_k}^\vee$$

$$(3.10) \quad = t_{-\lambda_-} s_{i_1} \dots s_{i_{k-1}} (-\alpha_{i_k}^\vee).$$

Recall that $t_{-\lambda_-}$ does not change the finite (bar) part of an affine coroot and that $s_{i_r} \dots s_{i_1} = w_0\sigma$. Therefore, the negated finite parts of $\beta_1^\vee, \dots, \beta_r^\vee$ are exactly the positive roots which are mapped to negative roots by $w_0\sigma$.

Recall the Orr-Shimozono formula ([OS], Proposition 5.4) for the $t = \infty$ specialization of the non-symmetric Macdonald polynomials ($\lambda'_- \in P$):

$$(3.11) \quad E_{\lambda'_-}(x, q^{-1}, \infty) = \sum_{p \in \overleftarrow{\mathcal{QB}}(id; u(\lambda'_-))} x^{wt(p)} q^{\deg(qwt^*(p))}.$$

Here we use the reduced decomposition $u(\lambda'_-) = \pi s_{i_{r+1}} \dots s_{i_M}$ and the corresponding coroots β_j . In other words, $E_{\lambda'_-}(x, q^{-1}, \infty)$ is equal to the sum over all paths in the reversed quantum Bruhat graph with $z_0 = u(\lambda'_-)$. We note that $\text{dir}(u(\lambda'_-)) = \text{dir}(t_{\lambda'_-} v(\lambda'_-)^{-1}) = w_0\sigma$. In order to pass from the reversed graph to the usual QBG we have to multiply by w_0 from the left. Hence the sum in (3.11) multiplied by w_0 from the left ranges over all paths in QBG starting at σ .

Corollary 3.16 gives the combinatorial formula of $\text{ch } W_{\sigma(\lambda_-)}(\ell(w_0) - \ell(\sigma))$, that is identical to $w_0 E_{w_0\sigma\lambda_-}(x, q^{-1}, \infty)$ by (3.11) through the above identification. Hence we obtain $\text{ch } W_{\sigma(\lambda_-)}(\ell(w_0) - \ell(\sigma)) = w_0 E_{w_0\sigma\lambda_-}(x, q^{-1}, \infty)$. \square

Corollary 3.19. *The character of $U_{\sigma(\lambda_-)}$ is equal to $w_0 E_{w_0\sigma(\lambda_-)}(x, q^{-1}, \infty)$. Equivalently, $E_{\sigma(\lambda_-)}(x, q^{-1}, \infty) = w_0 \text{ch } U_{w_0\sigma(\lambda_-)}$.*

Corollary 3.20. *The algebra $A_{(\lambda_-)_\sigma}$ acts freely on $\mathbb{U}_{\sigma(\lambda_-)}$ and*

$$\text{ch } \mathbb{U}_{\sigma(\lambda_-)} = \text{ch } U_{\sigma(\lambda_-)} / (q)_{(\lambda_-)_\sigma}.$$

Proof. Theorem 3.18 and its proof imply

$$\text{ch } \mathbb{U}_{\sigma(\lambda_-)} = \text{ch } \mathbb{W}_{\sigma(\lambda_-)}(\ell(w_0\sigma)) = \frac{\text{ch } \mathbb{W}_{\sigma(\lambda_-)}(\ell(w_0\sigma))}{(q)_\nu},$$

where ν is obtained from λ_- by adding all fundamental weights ω_j such that the corresponding simple roots α_j show up as $-\bar{\beta}_i^\vee$ for $i = 1, \dots, \ell(w_0) - \ell(\sigma)$ (see Theorem 3.17, (ii)). However, such α_j are exactly the simple roots mapped to Δ_+ by σ . We conclude that $\nu = (\lambda_-)_\sigma$. \square

4. GLOBAL U-MODULES AND SHEAVES ON SEMI-INFINITE SCHUBERT VARIETIES

Let \mathfrak{Q} be the semi-infinite flag variety (see [FiMi], [BF1]). For an element $w \in W$ we denote by $\mathfrak{Q}(w) \subset \mathfrak{Q}$ the corresponding semi-infinite

Schubert variety (see [Kat]). The varieties $\mathfrak{Q}(w)$ are defined as follows. Let $X(w) \in G/B$ be the (finite-dimensional) Schubert variety corresponding to the element w . Let $\text{ev}_0 : \mathfrak{Q}_0 \rightarrow G/B$ be the evaluation map from the subvariety $\mathfrak{Q}_0 \subset \mathfrak{Q}$ of no-defect quasi-maps to the flag variety of G . By definition, $\mathfrak{Q}(w) = \overline{\text{ev}^{-1}(X(w))}$. In particular, we have an embedding $X(w) \subset \mathfrak{Q}(w)$ consisting of constant loops. Let us denote the unique H -fixed point of the dense open B -orbit of $X(w)$ by x_w . We regard x_w as a point in $X(w) \subset \mathfrak{Q}(w)$.

Remark 4.1. The contents of this section can be also formulated by employing the formal model \mathbf{Q} of the semi-infinite flag variety (instead of the ind-model) defined in [FiMi, §4.1] by assuming the results from [BF1] and [KNS, §4].

We note that each semi-infinite Schubert variety inherits an ind-structure from \mathfrak{Q} , i.e. $\mathfrak{Q}(w) = \cup_{\beta \in Q_+^\vee} \mathfrak{Q}(w, \beta)$. Using the embedding

$$\mathfrak{Q}(w, \beta) \subset \prod_{i=1}^n \mathbb{P}(V(\omega_i) \otimes \mathbb{C}[z]_{\leq \langle \beta, \omega_i \rangle})$$

one gets for each $\lambda \in P$ the line bundle $\mathcal{O}_w(\lambda)$ on $\mathfrak{Q}(w)$ (this is the projective limit of the line bundles $\mathcal{O}_{w, \beta}(\lambda)$ on $\mathfrak{Q}(w, \beta)$). We define the i -th cohomology of $\mathcal{O}_w(\lambda)$ by

$$H^i(\mathfrak{Q}(w), \mathcal{O}_w(\lambda)) := \left(\varprojlim_{\beta} H^i(\mathfrak{Q}(w, \beta), \mathcal{O}_{w, \beta}(\lambda)) \right)_{\mathbb{G}_m\text{-finite}}.$$

It is proved in [Kat, Theorem 4.12] that for $\lambda \in P_+$ one has

$$(4.1) \quad H^0(\mathfrak{Q}(w), \mathcal{O}_w(\lambda))^* \simeq \mathbb{W}_{w\lambda},$$

where $*$ denotes the restricted dual and all the higher cohomologies vanish. Let us denote by $u_{w\lambda}$ the \mathfrak{I} -cyclic generator of $H^0(\mathfrak{Q}(w), \mathcal{O}_w(\lambda))^*$ that is fixed by the action of the loop rotation (such a vector is unique up to constant). In [Kat, §6], the author constructs sheaves $\mathcal{E}_w(\lambda)$ on $\mathfrak{Q}(w)$ such that

$$\text{ch } H^i(\mathfrak{Q}(w), \mathcal{E}_w(\lambda))^* = \delta_{i,0} E_{-w(\lambda)}(x^{-1}, q^{-1}, \infty) \in \mathbb{C}[P][[q]],$$

holds for each $\lambda \in P$, where x^{-1} means the replacement of e^μ by $e^{-\mu}$ for each $\mu \in P$. Moreover, $H^0(\mathfrak{Q}(w), \mathcal{E}_w(\lambda))^*$ is a cyclic \mathfrak{I} -module ([Kat, Lemma 6.7]).

Corollary 4.2. *For a dominant weight λ and $w \in W$ one has*

$$(4.2) \quad \text{ch } \Gamma(\mathfrak{Q}(w), \mathcal{E}_w(\lambda))^* = \text{ch } \mathbb{U}_{w\lambda}.$$

Proof. We set that $\sigma = ww_0$, and $\lambda_- = w_0\lambda$. Remark 3.6 implies $(q)_{\lambda_w} = (q)_{(\lambda_-)_\sigma}$. By [Kat, Corollary 6.10], we have an equality:

$$(4.3) \quad \text{ch } \Gamma(\mathfrak{Q}(w), \mathcal{E}_w(\lambda))^* = (q)_{\lambda_w}^{-1} \cdot E_{-w\lambda}(x^{-1}, q^{-1}, \infty).$$

We see that

$$E_{-w\lambda}(x^{-1}, q^{-1}, \infty) = w_0 E_{w_0 w \lambda}(x, q^{-1}, \infty) = w_0 E_{w_0 \sigma \lambda_-}(x, q^{-1}, \infty),$$

where the first equality is [OS, Lemma 5.2], and the second equality is by convention.

By Corollary 3.19, we have

$$\text{ch } U_{w\lambda} = \text{ch } U_{\sigma \lambda_-} = w_0 E_{w_0 \sigma \lambda_-}(x, q^{-1}, \infty).$$

Corollary 3.20 tells us that $\text{ch } \mathbb{U}_{\sigma \lambda_-} = (q)_{(\lambda_-)_\sigma}^{-1} \cdot \text{ch } U_{\sigma \lambda_-}$. Using (4.3) and Remark 3.6 we conclude that (4.2) holds true. \square

We briefly recall the construction of the sheaves $\mathcal{E}_w(\lambda)$ from [Kat]. Let $w = s_{i_1} \dots s_{i_l}$ be a reduced decomposition of w . Let $\mathbf{I}_k \supset \mathbf{I}$ be the parabolic subgroup corresponding to α_k that contains the Iwahori group $\mathbf{I} \subset G(\mathbb{C}(z))$. We define

$$\mathfrak{Q}(\mathbf{i}) = \mathbf{I}_{i_1} \times_{\mathbf{I}} \mathbf{I}_{i_2} \times_{\mathbf{I}} \dots \times_{\mathbf{I}} \mathbf{I}_{i_l} \times_{\mathbf{I}} \mathfrak{Q}(e),$$

where the last factor is the smallest semi-infinite Schubert variety corresponding to the identity element $e \in W$. We set

$$\gamma_1 := \alpha_{i_1}, \gamma_2 := s_{i_1} \alpha_{i_2}, \dots, \gamma_l := s_{i_1} s_{i_2} \dots s_{i_{l-1}} \alpha_{i_l}.$$

The roots γ_i are distinct to each other and each of them belongs to Δ_+ since our choice of \mathbf{i} is reduced. Note that if we have a subexpression \mathbf{i}' of \mathbf{i} , then we have natural embedding $\mathfrak{Q}(\mathbf{i}') \hookrightarrow \mathfrak{Q}(\mathbf{i})$ of the analogously defined variety by understanding that the elements from the missing factors i_j to be belong to $\mathbf{I} \subset \mathbf{I}_{i_j}$. This particularly induces an inclusion $\mathfrak{Q}(e) = \mathfrak{Q}(\emptyset) \hookrightarrow \mathfrak{Q}(\mathbf{i})$. Hence, we can regard x_e also as a point of $\mathfrak{Q}(\mathbf{i})$.

One has the multiplication map

$$q_{\mathbf{i}} : \mathfrak{Q}(\mathbf{i}) \rightarrow \mathfrak{Q}(w).$$

For each $1 \leq k \leq l = \ell(w)$, we consider the divisor $H_k \subset \mathfrak{Q}(\mathbf{i})$ defined by

$$H_k = \{(g_1, \dots, g_l, x) \in \mathfrak{Q}(\mathbf{i}), g_k \in \mathbf{I}\}.$$

Then the sheaf $\mathcal{E}_w(\lambda)$ on $\mathfrak{Q}(w)$ is obtained by twisting the line bundle corresponding to λ by the divisors H_k and pushing it down. Namely, we have

$$\mathcal{E}_w(\lambda) = (q_{\mathbf{i}})_* \mathcal{O}_{\mathfrak{Q}(\mathbf{i})}(\lambda - \sum_{j=1}^l H_j) = (q_{\mathbf{i}})_* \mathcal{O}_{\mathfrak{Q}(\mathbf{i})}(-\sum_{j=1}^l H_j) \otimes \mathcal{O}_w(\lambda).$$

We note that the sheaves $\mathcal{E}_w(\lambda)$ do not depend on the reduced decomposition of w ([Kat, Lemma 6.6]).

The maps $q_{\mathbf{i}}$ satisfy the following important properties ([Kat, Lemma 6.1 and Corollary 6.5]):

$$(4.4) \quad \mathbb{R}^k (q_{\mathbf{i}})_* \mathcal{O}_{\mathfrak{Q}(\mathbf{i})} = \delta_{k,0} \mathcal{O}_{\mathfrak{Q}(w)},$$

$$(4.5) \quad \mathbb{R}^k (q_{\mathbf{i}})_* \mathcal{O}_{\mathfrak{Q}(\mathbf{i})}(-\sum_{k=1}^l H_k) = 0, \quad k > 0.$$

We also note that the embedding

$$(q_i)_* \mathcal{O}_{\Omega(\mathbf{i})}(\lambda - \sum_{k=1}^l H_k) \subset (q_i)_* \mathcal{O}_{\Omega(\mathbf{i})}(\lambda)$$

gives the embedding $\mathcal{E}_w(\lambda) \hookrightarrow \mathcal{O}_{\Omega_w}(\lambda)$. Hence (4.1) yields an \mathfrak{I} -module surjection

$$(4.6) \quad \mathbb{W}_{w\lambda} \rightarrow \Gamma(\Omega(w), \mathcal{E}_w(\lambda))^*.$$

We conclude that the module $\Gamma(\Omega(w), \mathcal{E}_w(\lambda))^*$ is a cyclic \mathfrak{I} -module that is a quotient of the generalized global Weyl module $\mathbb{W}_{w\lambda}$.

Lemma 4.3. *There exists a surjection of \mathfrak{I} -modules*

$$\mathbb{U}_{w\lambda} \rightarrow H^0(\Omega(w), \mathcal{E}_w(\lambda))^*.$$

Proof. We write $w\lambda$ as $\sigma\lambda_-$ for $\sigma = ww_0$, and $\lambda_- = w_0\lambda$. Using the surjection (4.6), we only need to check that the relations

$$e_{\sigma(\alpha)}^{-\langle \lambda_-, \alpha^\vee \rangle} v = 0, \quad \alpha \in \Delta_+ \cap \sigma^{-1}\Delta_+$$

hold in $\Gamma(\Omega(w), \mathcal{E}_w(\lambda))^*$, where v is the image of $u_{w\lambda}$ (these are exactly the relations one has to add to the defining relations of $\mathbb{W}_{w\lambda}$ in order to get the module $\mathbb{U}_{w\lambda}$). Note that the $(H \times \mathbb{G}_m$ -eigen) dual vector of v (or $u_{w\lambda}$) corresponds to a constant function $1_{-w\lambda_-}$ on the dense \mathbf{I} -orbit of $\Omega(w)$.

We have an inclusion

$$N_{-\gamma_1} \times N_{-\gamma_2} \times \cdots \times N_{-\gamma_l} \times \mathbf{I} \subset \mathbf{I}_{i_1} \times_{\mathbf{I}} \mathbf{I}_{i_2} \times_{\mathbf{I}} \cdots \times_{\mathbf{I}} \mathbf{I}_{i_l},$$

where $N_{\pm\gamma_j}$ is the one-dimensional unipotent subgroup of $G(z)$ so that $\text{Lie } N_{\pm\gamma_j} \subset \hat{\mathfrak{g}}$ has \mathfrak{h} -weight $\pm\gamma_j$, respectively.

We consider a curve $\mathbb{P}_j^1 \subset \Omega(\mathbf{i})$ defined as the closure of the affine line $N_{-\gamma_j}x_e \subset \Omega(\mathbf{i})$. We refer this curve as \mathbb{P}_j^1 (it is isomorphic to \mathbb{P}^1). Since H and N_{γ_j} fixes x_e , it follows that \mathbb{P}_j^1 is equivariant with respect to the N_{γ_j} -action. Thus, \mathbb{P}_j^1 decomposes into the disjoint union of a point $\{x_e\}$ and a N_{γ_j} -orbit isomorphic to \mathbb{A}^1 .

Our curve $\mathbb{P}_j^1 \subset \Omega(\mathbf{i})$ is naturally contained in $\Omega(\{i_j, \dots, i_l\})$ so that $\mathbb{P}_j^1 \cap \Omega(\{i_{j+1}, \dots, i_l\}) = \{x_e\}$. Since $\mathcal{O}_{\Omega(\mathbf{i})}(\lambda)$ is determined by the H -character at x_e and is equivariant with respect to the group action, it follows that the restriction of $\mathcal{O}(\lambda)$ to \mathbb{P}_j^1 is $\mathcal{O}(m)$, where $m = -\langle \lambda_-, \alpha^\vee \rangle$. The restriction of H_k to \mathbb{P}_j^1 is non-zero if and only if $j = k$, and it defines $\mathcal{O}(1)$ when $j = k$.

Therefore, we restrict the sheaves $\mathcal{O}_{\Omega(\mathbf{i})}(\lambda)$ and $\mathcal{O}_{\Omega(\mathbf{i})}(\lambda - \sum_{k=1}^l H_k)$ to \mathbb{P}_j^1 to obtain the following maps:

$$\begin{aligned} H^0(\Omega(\mathbf{i}), \mathcal{O}_{\Omega(\mathbf{i})}(\lambda)) &\rightarrow H^0(\mathbb{P}_j^1, \mathcal{O}(m)), \\ H^0(\Omega(\mathbf{i}), \mathcal{O}_{\Omega(\mathbf{i})}(\lambda - \sum_{k=1}^l H_k)) &\rightarrow H^0(\mathbb{P}_j^1, \mathcal{O}(m-1)). \end{aligned}$$

These maps are equivariant with respect to the N_{γ_j} -action. The former map is non-zero since $1_{-w\lambda_-}$ induces a non-vanishing section of both of them. The section $1_{-w\lambda_-}$ also induces an \mathfrak{J} -cocyclic vector of $H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda))$ and a N_{γ_j} -cocyclic vector in $H^0(\mathbb{P}^1, \mathcal{O}(m))$. By using the embedding $\mathcal{O}(\lambda - \sum_{k=1}^l H_k) \hookrightarrow \mathcal{O}(\lambda)$ and dualizing all the pieces, we obtain the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^1, \mathcal{O}(m))^* & \longrightarrow & H^0(\mathbb{P}^1, \mathcal{O}(m-1))^* \\ \kappa \downarrow & & \downarrow \kappa' \\ \mathbb{W}_{w(\lambda)} & \longrightarrow & H^0(\mathfrak{Q}(w), \mathcal{E}_w(\lambda))^* \end{array}$$

from (4.1) and (4.5). Here all the spaces have common cyclic vector (with respect to the N_{γ_j} -action in the top line, and with respect to the \mathfrak{J} -action in the bottom line) induced by $u_{w\lambda}$.

Note that $U(\mathfrak{b})u_{w\lambda} \subset \mathbb{W}_{w(\lambda)} \subset \mathbb{W}_\lambda$ spans a Demazure submodule of \mathfrak{g} . In particular, the span of $\{e_{\gamma_j}^n u_{w\lambda}\}_{n \geq 0}$ constitutes a representation of $\mathfrak{sl}(2)$ corresponding to the (not necessarily simple) roots $\pm \gamma_j$. In particular, we deduce that

$$e_{\gamma_j}^m u_{w\lambda} \neq 0 \quad \text{and} \quad e_{\gamma_j}^{m+1} u_{w\lambda} = 0.$$

This implies that the map κ is injective.

By constriction, the N_{γ_j} -cyclic H -eigenvector of $H^0(\mathbb{P}^1, \mathcal{O}(m-1))^*$ is annihilated by $e_{\gamma_j}^m$ as the corresponding cyclic vector is annihilated by $e_{\gamma_j}^{m+1}$ in $H^0(\mathbb{P}^1, \mathcal{O}(m))^*$. Sending it through κ' , the above commutative diagram asserts that $e_{\gamma_j}^m v = 0$. This proves our Lemma. \square

Recall that the star multiplication on W is defined by $s_i * w = s_i w$ if $\ell(s_i w) = \ell(w) + 1$ and $s_i * w = w$ otherwise ($i \in I$). This makes $(W, *)$ into a monoid. For each $k = 1, \dots, l$, let $w[k] = s_{i_1} * \dots * s_{i_{k-1}} * s_{i_{k+1}} * \dots * s_{i_l}$.

Lemma 4.4. *Consider the embedding*

$$\varphi_k : H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda - H_k)) \hookrightarrow H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda)).$$

Then $\ker \varphi_k^ \subset H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda))^* \simeq \mathbb{W}_{w\lambda}$ is equal to $\mathbb{W}_{w[k]\lambda}$.*

Proof. For any $k = 1, \dots, l$, we have the following exact sequence of sheaves:

$$0 \rightarrow \mathcal{O}_{\mathfrak{Q}(\mathbf{i})}(-H_k) \rightarrow \mathcal{O}_{\mathfrak{Q}(\mathbf{i})} \rightarrow \mathcal{O}_{\mathfrak{Q}(\mathbf{i}')} \rightarrow 0,$$

where \mathbf{i}' corresponds to omitting i_k in $\mathbf{i} = \{i_j\}_j$. Note that

$$H^i(\mathfrak{Q}(\mathbf{i}'), \mathcal{O}_{\mathfrak{Q}(\mathbf{i}')}(\lambda)) = H^i(\mathfrak{Q}(\mathbf{i}), \mathcal{O}_{\mathfrak{Q}(\mathbf{i}')}(\lambda))$$

for each $i \geq 0$ as $\mathfrak{Q}(\mathbf{i}') \subset \mathfrak{Q}(\mathbf{i})$ is a closed (ind-)subvariety. In view of [Kat, Proposition 6.4], we apply $H^0(\cdot)^*$ to obtain

$$0 \rightarrow H^0(\mathfrak{Q}(\mathbf{i}'), \mathcal{O}_{\mathfrak{Q}(\mathbf{i}')}(\lambda))^* \rightarrow H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}_{\mathfrak{Q}(\mathbf{i}')}(\lambda))^* \rightarrow H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}_{\mathfrak{Q}(\mathbf{i})}(\lambda - H_k))^* \rightarrow 0.$$

By [Kat, Lemma 6.1] and the fact that the multiplication map of $\mathfrak{Q}(\mathbf{i}') \subset \mathfrak{Q}(\mathbf{i})$ lands exactly on $\mathfrak{Q}(w[k])$, we deduce

$$H^0(\mathfrak{Q}(\mathbf{i}'), \mathcal{O}_{\mathfrak{Q}(\mathbf{i}')}(\lambda))^* \cong H^0(\mathfrak{Q}(w[k]), \mathcal{O}_{\mathfrak{Q}(w[k])}(\lambda))^* \cong \mathbb{W}_{w[k]\lambda}$$

as required. \square

Theorem 4.5. *The \mathfrak{I} -modules surjection $\mathbb{U}_{w\lambda} \rightarrow \Gamma(\mathfrak{Q}(w), \mathcal{E}_w(\lambda))^*$ is an isomorphism.*

Proof. We have the following equality, where all the spaces are considered as subspaces of $H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda))$:

$$H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda - \sum_{k=1}^l H_k)) = \bigcap_{k=1}^l H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda - H_k)).$$

We conclude that

$$H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda - \sum_{k=1}^l H_k))^* \simeq \frac{H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda))^*}{\sum_{k=1}^l \ker \varphi_k^*}.$$

So, theorem follows from the equality

$$(4.7) \quad \sum_{k=1}^l \mathbb{W}_{w[k]\lambda} = \sum_{\alpha \in \Delta_+ \cap \sigma^{-1}\Delta_+} \mathbb{W}_{s_{\sigma(\alpha)}w\lambda},$$

where all the spaces are considered as subspaces of $\mathbb{W}_{w\lambda}$ (that is isomorphic to $H^0(\mathfrak{Q}(\mathbf{i}), \mathcal{O}(\lambda))^*$). Indeed, one has

$$\mathbb{U}_{w\lambda} = \frac{\mathbb{W}_{w\lambda}}{\sum_{\alpha \in \Delta_+ \cap \sigma^{-1}\Delta_+} \mathbb{W}_{s_{\sigma(\alpha)}w\lambda}}.$$

We prove (4.7) in a separate lemma below. \square

Lemma 4.6. *Let λ be a dominant weight. Then for any element $w = \sigma w_0 \in W$ so that $\ell(w) = l$, we have the following equality of the subspaces of $\mathbb{W}_{w\lambda}$:*

$$\sum_{k=1}^l \mathbb{W}_{w[k]\lambda} = \sum_{\alpha \in \Delta_+ \cap \sigma^{-1}\Delta_+} \mathbb{W}_{s_{\sigma(\alpha)}w\lambda},$$

Proof. We first rewrite the left hand side. We take the maximal modules among the summands and conclude that the left hand side is equal to the sum over such $k = 1, \dots, l$ such that $\ell(w[k]) = l - 1$ (i.e. after removing the k -th factor in the reduced decomposition of w we still obtain a reduced expression). This is equivalent to saying that the left hand side is equal to the sum of the global generalized Weyl modules $\mathbb{W}_{s_\gamma w\lambda}$ such that there exists an edge $w^{-1}s_\gamma \rightarrow w^{-1}$ in the classical Bruhat graph.

Now let us consider the right hand side. Taking the maximal summands, we only consider α such that there exists an edge $\sigma \rightarrow \sigma s_\alpha$ in the classical Bruhat graph (and we still have $\sigma(\alpha) \in \Delta_+$). Now let $\gamma = \sigma(\alpha)$. Then the right hand side is equal to the sum of the global generalized Weyl modules

$\mathbb{W}_{s_\gamma w\lambda}$ such that there is an edge (recall $w = \sigma w_0$) from ww_0 to $s_\gamma ww_0$. Now taking inverse elements and multiplying by w_0 , we obtain that the summands correspond to the edges $w^{-1}s_\gamma \rightarrow w^{-1}$ in the classical Bruhat graph. This proves the lemma. \square

5. DUALITY OF LOCAL AND GLOBAL MODULES

Throughout this section, we assume that \mathfrak{g} is of type ADE. In particular, W^a is the affine Weyl group of \mathfrak{g} . We extend the integral weight lattice P of \mathfrak{g} to a weight lattice P^a of the untwisted affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ corresponding to the simple Lie algebra \mathfrak{g} as

$$P^a := P \oplus \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\delta,$$

where we regard $P \oplus \mathbb{Z}\delta$ as the set of level zero integral weights, and Λ_0 is the level one basic fundamental weight, and δ is the primitive null-root. We denote by $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$, and denote by α_0 the affine simple root of $\widehat{\mathfrak{g}}$ (with its coroot α_0^\vee). Let $s_0 \in W^a$ be the simple reflection corresponding to α_0 . We set $I^a := \{0, 1, \dots, n\}$. We have

$$\{\alpha_i\}_{i \in I} \subset \{\alpha_i\}_{i \in I^a} \subset P^a \subset \widehat{\mathfrak{h}}^*.$$

We have a reduced expression

$$(5.1) \quad u(\lambda) = s_{i_1} s_{i_2} \dots s_{i_\ell} \pi,$$

where π is a length zero element in the affine Weyl group. Let $\Lambda := \pi\Lambda_0$. We have a level one integrable highest weight representation $L(\Lambda)$ associated to Λ , and $u(\lambda)$ defines a Demazure submodule D_λ of $L(\Lambda)$ corresponding to $u(\lambda)\pi^{-1} \in W^a$. By its definition, D_λ is a finite-dimensional $\widehat{\mathfrak{h}}$ -semisimple \mathfrak{J} -module. Moreover, it has a cyclic vector of weight $\lambda + \Lambda_0$ by our type ADE assumption.

In addition, we regard the module $\mathbb{U}_{-\lambda}$ as a module whose cyclic vector has weight $-\lambda - \Lambda_0$ (that is possible as the defining equation as $U(\mathfrak{n}^{af})$ -modules completely determines the structure of $\mathbb{U}_{-\lambda}$ up to $\widehat{\mathfrak{h}}$ -weight twists; see Remark 2.1).

Theorem 5.1 (Sanderson-Ion [S, I]). *We have $\text{ch } D_\lambda = E_\lambda(x, q, 0)$.*

Let \mathfrak{B} be the category of $U(\mathfrak{J})$ -modules M such that M is semi-simple with respect to the $\widehat{\mathfrak{h}}$ -action with each $\widehat{\mathfrak{h}}$ -weight space is at most countable dimension, and its weights belong to P^a . In particular, every module M in \mathfrak{B} admits a \mathbb{Z} -grading coming from the $\mathbb{Z}\delta$ -part of the weight lattice (corresponding to the eigenvalues of the grading operator $d \in \widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$). In particular, \mathfrak{B} is a graded abelian category.

Let \mathfrak{B}' be the fullsubcategory of \mathfrak{B} so that each $\widehat{\mathfrak{h}}$ -weight space is finite dimensional, and its weights belong to $\Lambda + \sum_{i \in I^a} \mathbb{Z}_{\geq 0} \alpha_i$ for some $\Lambda \in P^a$. Let $\mathfrak{B}_0 \subset \mathfrak{B}$ be the fullsubcategory consisting of finite-dimensional modules in \mathfrak{B} . The both \mathfrak{B}' and \mathfrak{B}_0 are graded abelian categories.

Lemma 5.2. *The category \mathfrak{B} has enough projectives.*

Proof. By $\widehat{\mathfrak{h}}$ -semisimplicity, the maximal cyclic \mathfrak{I} -module in \mathfrak{B} that surjects onto \mathbb{C}_Λ is $U(\mathfrak{I}) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\Lambda$. By the Frobenius reciprocity, this module maps to every module in \mathfrak{B} that has non-zero weight Λ -part. Collecting them for all weights, we obtain a surjection from a projective module to an arbitrary module in \mathfrak{B} as required. \square

Let $\Lambda \in P^a$. We denote by \mathbb{C}_Λ the one-dimensional \mathfrak{I} -module whose action factors through

$$(\mathfrak{I} + \widehat{\mathfrak{h}}) \rightarrow (\mathfrak{I} + \widehat{\mathfrak{h}})/[\mathfrak{I}, \mathfrak{I}] \cong \widehat{\mathfrak{h}} \xrightarrow{\Lambda} \mathbb{C}.$$

Since $[\mathfrak{I}, \mathfrak{I}]$ is a (pro-)nilpotent Lie algebra, it follows that $\{\mathbb{C}_\Lambda\}_{\Lambda \in P^a}$ is the complete collection of simple modules in \mathfrak{B} . Let P_Λ be the projective cover of \mathbb{C}_Λ in \mathfrak{B} .

Proposition 5.3 (see e.g. Kumar [Kum] Chapter III). *For each $\Lambda \in P^a$, we have*

$$(5.2) \quad \text{ch } P_\Lambda = \prod_{\alpha \in \Delta_+^a} (1 - e^\alpha)^{-1} \cdot \text{ch } \mathbb{C}_\Lambda.$$

Proof. Projective modules in \mathfrak{B} are isomorphic to $U(\mathfrak{I}) \otimes_{U(\widehat{\mathfrak{g}})} \mathbb{C}_\Lambda$. Hence the Poincaré-Birkoff-Witt theorem applied to $U([\mathfrak{I}, \mathfrak{I}])$ gives its character. \square

For each $i \in I^a$ and a $U(\mathfrak{I})$ -module M , we define $\mathcal{D}_i(M)$ to be the maximal $\mathfrak{sl}(2, i)$ -integrable quotient of $U(\mathfrak{I}_i) \otimes_{U(\mathfrak{I})} M$.

Lemma 5.4. *For each $i \in I^a$, the functor \mathcal{D}_i preserves \mathfrak{B} .*

Proof. For $M \in \mathfrak{B}$, the $U(\mathfrak{I})$ -module

$$U(\mathfrak{sl}(2, i) + \mathfrak{I}) \otimes_{U(\mathfrak{I})} M \cong U(\mathfrak{sl}(2, i)) \otimes_{U(\mathfrak{I} \cap \mathfrak{sl}(2, i))} M$$

sits in \mathfrak{B} . Therefore, its quotient $\mathcal{D}_i M$ also lie in \mathfrak{B} as required. \square

Theorem 5.5 (Joseph [J]). *The functors $\{\mathcal{D}_i\}_{i \in I^a}$ satisfy:*

- Each \mathcal{D}_i is right exact;
- We have a natural transformation $\text{Id} \rightarrow \mathcal{D}_i$;
- For two $i, j \in I$ so that $(s_i s_j)^m = 1$, we have

$$\overbrace{\mathcal{D}_i \mathcal{D}_j \cdots}^m \cong \overbrace{\mathcal{D}_j \mathcal{D}_i \cdots}^m;$$

- We have $\mathcal{D}_i^2 \cong \mathcal{D}_i$.

Proof. We warn that Joseph's original formulation is for semi-simple Lie algebra, but the identical proof works for Kac-Moody algebras. The first two assertions are [J, Lemma 2.2]. The third assertion is [J, Proposition 2.15]. Note that the functorial isomorphism in the third assertion follows from the fact that the resulting functor yields a direct sum of finite-dimensional representations of simple Lie algebra generated by $e_{\alpha_i}, e_{\alpha_j}, f_{\alpha_i}, f_{\alpha_j}$. The

fourth assertion follows as \mathcal{D}_i does not change a module that is $\mathfrak{sl}(2, \alpha_i)$ -integrable. \square

Let $D^-(\mathfrak{B})$ be the derived category of \mathfrak{B} bounded from the below. The restricted dual $*$ induces an endo-functor on the fullsubcategory of \mathfrak{B} -modules whose weight spaces are finite-dimensional. It perserves \mathfrak{B}_0 . We set $\mathcal{D}_i^\dagger := * \circ \mathcal{D}_i \circ *$ for each $i \in I^a$. Let $\mathbb{L}\mathcal{D}_i$ (resp. $\mathbb{R}\mathcal{D}_i^\dagger$) be the left derived functor of \mathcal{D}_i (resp. the $*$ -conjugation of $\mathbb{L}\mathcal{D}_i$). The functor $\mathbb{R}\mathcal{D}_i^\dagger$ lands on \mathfrak{B}_0 thanks to the following:

Lemma 5.6. *Let $i \in I^a$. For each $N \in \mathfrak{B}_0$ and $k \in \mathbb{Z}$, we have*

$$H^k(\mathbb{R}\mathcal{D}_i^\dagger(N)) \cong H^k(\mathbb{P}^1, \mathcal{O}(N)),$$

where $\mathcal{O}(N)$ is the $SL(2, \mathbb{C})$ -equivariant vector bundle on \mathbb{P}^1 obtained from N . In particular, the total cohomology of $\mathcal{D}_i^\dagger N$ lies in \mathfrak{B}_0 .

Proof. We have a functorial isomorphism (with respect to $N \in \mathfrak{B}_0$)

$$(5.3) \quad (U(\mathfrak{sl}(2, i)) \otimes_{U(\mathfrak{J} \cap \mathfrak{sl}(2, i))} N^*)^* \cong H^0(U, \mathcal{O}(N)) \cong \mathbb{C}[\mathbb{A}] \otimes N,$$

where $\mathbb{A} \subset \mathbb{P}^1$ denotes the open dense \mathbf{I} -orbit of $\mathbb{P}^1 \cong \mathbf{I}_i/\mathbf{I}$. The maximal $\mathfrak{sl}(2, i)$ -finite submodule of the LHS of (5.3) is $H^0(\mathbb{R}\mathcal{D}_i^\dagger(N))$, and the maximal $\mathfrak{sl}(2, i)$ -finite submodule of the RHS of (5.3) is $H^0(\mathbb{P}^1, \mathcal{O}(N))$. By construction, the both of $\{H^k(\mathbb{R}\mathcal{D}_i^\dagger(\bullet))\}_k$ and $\{H^k(\mathbb{P}^1, \mathcal{O}(\bullet))^*\}_k$ are the universal δ -functors (as it is straight-forward to check that some finite-dimensional submodule of the injective envelope yields an effacable envelope, see [G, §2.1–2.2]). Being universal δ -functors of two isomorphic functors, they are necessarily isomorphic as desired. \square

Proposition 5.7. *Let $i \in I^a$. For $M \in \mathfrak{B}$ and $N \in \mathfrak{B}_0$, we have*

$$\mathrm{ext}_{\mathfrak{B}}^k(\mathbb{L}\mathcal{D}_i(M), N) \cong \mathrm{ext}_{\mathfrak{B}}^k(M, \mathbb{R}\mathcal{D}_i^\dagger(N)) \quad k \in \mathbb{Z},$$

where ext are understood as the hypercohomologies.

Proof. We set $\alpha := \alpha_i$. Let us denote by $\mathfrak{b}_0 := \widehat{\mathfrak{h}} \oplus E_\alpha = (\mathfrak{J} \cap (\widehat{\mathfrak{h}} + \mathfrak{sl}(2, i)))$ and $\mathfrak{g}_0 := \widehat{\mathfrak{h}} + \mathfrak{sl}(2, i)$. Let us denote by $V_0(\Lambda)$ be the irreducible finite-dimensional $(\mathfrak{g}_0 + \widehat{\mathfrak{h}})$ -module with highest weight $\Lambda \in P^a$. For each $\Lambda \in P^a$, we have

$$\mathcal{D}_i(U(\mathfrak{J}) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\Lambda) \cong \begin{cases} \bigoplus_{n \geq 0} U(\mathfrak{J}) \otimes_{U(\mathfrak{b}_0)} V_0(\Lambda + n\alpha) & (\langle \alpha^\vee, \Lambda \rangle \geq 0) \\ \bigoplus_{n \geq 0} U(\mathfrak{J}) \otimes_{U(\mathfrak{b}_0)} V_0(s_i \Lambda + n\alpha) & (\langle \alpha^\vee, \Lambda \rangle < 0) \end{cases}.$$

Let us consider a $\widehat{\mathfrak{h}}$ -semisimple indecomposable \mathfrak{b}_0 -module $N_{\Gamma, m}$ with lowest weight $\Gamma \in P^a$ and highest weight $\Gamma + m\alpha$ (such a module is unique up to isomorphism, cf. [J, §2.3]). We have

$$\mathcal{D}_i^\dagger(N_{\Gamma, m}) \cong \begin{cases} \bigoplus_{0 \leq n \leq \min\{m, -m - \langle \alpha^\vee, \Gamma \rangle\}} V_0(s_i \Gamma - n\alpha) & (m \leq -\langle \alpha^\vee, \Gamma \rangle) \\ \{0\} & (m > -\langle \alpha^\vee, \Gamma \rangle) \end{cases}.$$

By the Frobenius-Nakayama reciprocity, we have

$$(5.4) \quad \text{ext}_{\mathfrak{B}}^k(U(\mathfrak{J}) \otimes_{U(\widehat{\mathfrak{h}})} M, N) \cong \text{ext}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}^k(U(\mathfrak{b}_0) \otimes_{U(\widehat{\mathfrak{h}})} M, N),$$

for a finitely generated $U(\mathfrak{b}_0)$ -module M with semi-simple $\widehat{\mathfrak{h}}$ -action and $N \in \mathfrak{B}_0$, where the RHS denotes the relative extension (cf. Kumar [Kum, Chapter III]).

In view of [J, §2.3], it suffices to compute the extension by replacing $U(\mathfrak{J}) \otimes_{U(\widehat{\mathfrak{h}})} M$ with $\mathbb{L}\mathcal{D}_i(U(\mathfrak{b}_0) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\Lambda)$ and N with a string $U(\mathfrak{b}_0)$ -module to see the desired isomorphism for a projective module M and a finite-dimensional module N .

In other words, our assertion reduces to the functorial isomorphism:

$$(5.5) \quad \text{ext}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}^k(\mathbb{L}\mathcal{D}_i(U(\mathfrak{b}_0) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\Lambda), N_{\Gamma, m}) \cong \text{ext}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}^k(U(\mathfrak{b}_0) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\Lambda, \mathbb{R}\mathcal{D}_i^\dagger(N_{\Gamma, m})),$$

for $\Lambda, \Gamma \in P^a$, $m \in \mathbb{Z}_{\geq 0}$, and $k \in \mathbb{Z}$, where \mathcal{D}_i and \mathcal{D}_i^\dagger are replaced with analogous functors to \mathcal{D}_i and \mathcal{D}_i^\dagger defined for \mathfrak{b}_0 -modules with semi-simple $\widehat{\mathfrak{h}}$ -actions. Our functors in (5.5) are universal δ -functors as $\mathbb{L}\bullet\mathcal{D}_i$ is co-effaceable by taking projective cover, and $\mathbb{R}\bullet\mathcal{D}_i^\dagger$ is effaceable by taking a finite-dimensional submodule inside its injective envelope (cf. [G, §2.1–2.2]). Therefore, it suffices to prove (5.5) for $k = 0$.

Here the $k = 0$ case of the RHS of (5.5) further reduces to

$$\begin{aligned} \text{hom}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}(U(\mathfrak{b}_0) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\Lambda, \mathcal{D}_i^\dagger(N_{\Gamma, m})) &\cong \text{hom}_{\widehat{\mathfrak{h}}}(\mathbb{C}_\Lambda, \mathcal{D}_i^\dagger(N_{\Gamma, m})) \\ &\cong \begin{cases} \bigoplus_{0 \leq n \leq n_0} \text{hom}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}(\mathbb{C}_\Lambda, V_0(s_i \Gamma - n\alpha)) & (m \leq -\langle \alpha^\vee, \Gamma \rangle) \\ \{0\} & (m > -\langle \alpha^\vee, \Gamma \rangle) \end{cases} \end{aligned}$$

by the Frobenius reciprocity, where $n_0 := \min\{m, -m - \langle \alpha^\vee, \Gamma \rangle\}$. The $k = 0$ case of LHS of (5.5) is rephrased as:

$$\begin{aligned} \text{hom}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}(\mathcal{D}_\alpha(U(\mathfrak{b}_0) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\Lambda), N_{\Gamma, m}) \\ \cong \begin{cases} \bigoplus_{n \geq 0} \text{hom}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}(V_0(\Lambda + n\alpha), N_{\Gamma, m}) & (\langle \alpha^\vee, \Lambda \rangle \geq 0) \\ \bigoplus_{n \geq 0} \text{hom}_{(\mathfrak{b}_0, \widehat{\mathfrak{h}})}(V_0(s_i \Lambda + n\alpha), N_{\Gamma, m}) & (\langle \alpha^\vee, \Lambda \rangle < 0) \end{cases} \end{aligned}$$

From these, we derive the desired isomorphisms (5.5). Moreover, these isomorphisms are functorial with respect to the morphism of modules as it commutes with the morphisms in each variable.

Therefore, we conclude the desired functorial isomorphism as required. \square

Below in this section, every functor is derived unless stated otherwise.

Lemma 5.8 ([Kat] §4, particularly Theorem 4.13). *For each $i \in I$ and $\lambda \in P$, we have*

$$\mathcal{D}_i(\mathbb{W}_\lambda) \cong \begin{cases} \mathbb{W}_{s_i \lambda} & (\langle \alpha_i^\vee, \lambda \rangle > 0) \\ \mathbb{W}_\lambda & (\langle \alpha_i^\vee, \lambda \rangle \leq 0) \end{cases}.$$

Proposition 5.9. *For each weight λ and $i \in I^a$, we have*

$$(5.6) \quad H_k(\mathcal{D}_i(D_\lambda)) \cong \begin{cases} D_{s_i\lambda} & (k=0, u(\lambda) < s_i u(\lambda) \notin u(\lambda)W) \\ D_\lambda & (k=0, u_\lambda > s_i u(\lambda) \text{ or } s_i u(\lambda) \in u(\lambda)W) \\ \{0\} & (k \neq 0) \end{cases}$$

Assume that \mathfrak{g} is not of type $E_8F_4G_2$. If we have $s_i u(\lambda) < u(\lambda)$, then we have a short exact sequence

$$0 \rightarrow \mathbb{U}_{-\lambda} \rightarrow \mathcal{D}_i(\mathbb{U}_{-\lambda}) \rightarrow \mathbb{U}_{-s_i\lambda} \rightarrow 0.$$

If we have $s_i u(\lambda) \in u(\lambda)W$, then we have $\mathbb{U}_{-\lambda} \cong \mathcal{D}_i(\mathbb{U}_{-\lambda})$. If we have $u(\lambda) < s_i u(\lambda) \notin u(\lambda)W$, then we have $\mathcal{D}_i(\mathbb{U}_{-\lambda}) \cong \{0\}$. In each case, we have $H_k(\mathcal{D}_i(\mathbb{U}_{-\lambda})) \cong \{0\}$ for $k \neq 0$.

Proof. The first assertion is a rephrasement of [Kum, Theorem 8.2.2 and Theorem 8.2.9] applied to $\pi\Lambda_0 \in P^a$ and $u(\lambda)\pi^{-1} \in W^a$.

We prove the second assertion. In view of the construction of [Kat, §6] (see also §4) and Lemma 5.6, we deduce a short exact sequence

$$(5.7) \quad 0 \rightarrow \mathbb{U}_{-\lambda} \rightarrow \mathcal{D}_i(\mathbb{U}_{-\lambda}) \rightarrow \mathbb{U}_{-s_i\lambda} \rightarrow 0$$

when $s_i u(\lambda) < u(\lambda)$. Moreover, the corresponding higher cohomologies must vanish. By Theorem 5.5 4), it holds that applying \mathcal{D}_i to (5.7) yields an isomorphism $\mathcal{D}_i(\mathbb{U}_{-\lambda}) \rightarrow \mathcal{D}_i^2(\mathbb{U}_{-\lambda})$. Hence, we have $\mathcal{D}_i(\mathbb{U}_{-s_i\lambda}) \cong \{0\}$ by the associated long exact sequence.

We consider the case $s_i u(\lambda) \in u(\lambda)W$. We have $\lambda = w\lambda_-$ for $w \in W$. In view of the last formula in the proof of Theorem 4.5 and Lemma 4.6, we know that $\mathbb{U}_{-w\lambda_-}$ is a quotient of $\mathbb{W}_{-w\lambda_-}$ by $\mathbb{W}_{-u\lambda_-}$ with all $u < w \in W$. Here, we have $s_\alpha\lambda = \lambda$, which implies that $\mathbb{W}_{-w\lambda_-} = \mathbb{W}_{-s_i w\lambda_-}$ and $s_i w \cdot \text{stab}_W \lambda_- = w \cdot \text{stab}_W \lambda_-$. Note that w can be thought of as a minimal length element in $w \cdot \text{stab}_W \lambda_- \subset W$. It follows that $s_i u \notin w \cdot \text{stab}_W \lambda_-$ and $s_i u < s_i w$. It implies

$$\mathbb{W}_{-s_i u\lambda_-} \subsetneq \mathbb{W}_{-s_i w\lambda_-} = \mathbb{W}_{-w\lambda_-}.$$

Therefore, we have necessarily $\mathbb{W}_{-v\lambda_-} = \mathbb{W}_{-s_i u\lambda_-}$ with $v < w$ in view of [Kat, Theorem 4.12 (2)]. This implies $\mathcal{D}_i(\mathbb{U}_{-\lambda}) \cong \mathbb{U}_{-\lambda}$ by Lemma 5.8 and the last formula in the proof of Theorem 4.5.

In case $i = 0$, then we apply a diagram automorphism τ of the affine Dynkin diagrams of type $ABCDE_6E_7$ to α_0 and $\mathbb{U}_{-\lambda}$. Then, $\tau\alpha_0 = \alpha_i$ for some $i \in I$, and $\tau(\lambda + \Lambda_0) = \lambda' + \Lambda_0$ for some $\lambda' \in P$ so that $\langle \alpha_i^\vee, \lambda' \rangle < 0$. In addition, we have $\tau(\mathbb{U}_{-\lambda}) \cong \mathbb{U}_{-\lambda'}$ by the description of the defining equations. Therefore, we deduce the assertion also in this case. \square

Lemma 5.10. *Suppose that λ is anti-dominant. The module \mathbb{W}_λ admits a resolution by $\{P_\gamma \langle m \rangle\}_{\gamma, m \in \mathbb{Z}}$, where $w\gamma < \lambda$ for some $w \in W$ or $\gamma = \lambda$ (up to $\mathbb{Z}\Lambda_0$ -character twists).*

Proof. By a result of Chari-Ion [CI], we deduce that \mathbb{W}_λ admits a resolution by $U(\mathfrak{g}[z]) \otimes_{U(\mathfrak{g})} V(w_0\gamma)$ (that is a projective module in the category of

\mathfrak{g} -integrable $\mathfrak{g}[z]$ -modules, see [CG]), where $\gamma \leq \lambda$. Since $V(w_0\gamma)$ admits a finite resolution by $\{U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} \mathbb{C}_{w(\gamma-\rho)+\rho}\}_{w \in W}$ (afforded by the BGG resolution) as $U(\mathfrak{g})$ -modules, we deduce that the total complex of the double complex resolving each $U(\mathfrak{g}[z]) \otimes_{U(\mathfrak{g})} V(w_0\gamma)$ (direct summand of a term in the projective resolution in the category of \mathfrak{g} -integrable $\mathfrak{g}[z]$ -modules) has $\{P_{w(\gamma-\rho)+\rho}\}_{\gamma, w \in W}$ (γ is as above) as its direct factors. Since we have

$$w^{-1}(w(\gamma - \rho) + \rho) \leq \gamma,$$

and the equality holds if and only if $w = e$, we conclude the assertion. \square

Lemma 5.11. *Assume that \mathfrak{g} is of type ADE. Suppose that $\mu \in P_-$. The module D_μ admits a resolution by $\{P_\gamma \langle m \rangle\}_{\gamma, m \in \mathbb{Z}}$, where $\gamma \in P$ satisfies $w\gamma < \mu$ for some $w \in W$ or $\gamma = \mu$ (up to $\mathbb{Z}\Lambda_0$ -character twists).*

Proof. The module W_μ admits a finite resolution by a complex whose terms are the direct sum of \mathbb{W}_μ (since \mathbb{W}_μ admits an action of a polynomial ring and its specialization to a point is W_μ by [FL, N]). Hence, Lemma 5.10 implies that W_μ admits a $U(\mathfrak{J})$ -module resolution of the desired type. Therefore, the identification $D_\mu \cong W_\mu$ (see Remark 3.4) implies the result. \square

Theorem 5.12. *Assume that \mathfrak{g} is of type ADE_6E_7 . We have:*

$$\text{ext}_{\mathfrak{B}}^i(\mathbb{U}_{-\lambda}, D_\mu^*) \cong \begin{cases} \mathbb{C} & (i = 0, \lambda = \mu) \\ \{0\} & (\text{otherwise}). \end{cases}$$

Proof. If $\lambda - \mu \notin Q$, then the extension trivially vanish.

If we have $i \in I^a$ so that $s_i u(\mu) < u(\mu)$ or $s_i u_\mu \in u(\mu)W$ and $u(\lambda) < s_i u(\lambda) \notin u(\lambda)W$, then we have

$$\begin{aligned} \text{ext}_{\mathfrak{B}}^\bullet(\mathbb{U}_{-\lambda}, D_\mu^*) &\cong \text{ext}_{\mathfrak{B}}^\bullet(\mathbb{U}_{-\lambda}, \mathcal{D}_i^\dagger(D_\mu^*)) \\ &\cong \text{ext}_{\mathfrak{B}}^\bullet(\mathcal{D}_i(\mathbb{U}_{-\lambda}), D_\mu^*) \\ &\cong \text{ext}_{\mathfrak{B}}^\bullet(\{0\}, D_\mu^*) = \{0\}. \end{aligned}$$

This particularly implies

$$\text{ext}_{\mathfrak{B}}^\bullet(\mathbb{U}_{-\lambda}, D_\mu^*) = \{0\}$$

whenever there exists $i \in I$ so that $\langle \alpha_i^\vee, \lambda \rangle > 0 \geq \langle \alpha_i^\vee, \mu \rangle$ or $\langle \vartheta^\vee, \lambda \rangle \leq 0 < \langle \vartheta^\vee, \mu \rangle$.

Assume that λ and μ are both anti-dominant. Applying Lemma 5.11, we obtain an injective resolution of D_μ^* as $U(\mathfrak{J})$ -module whose simple submodules are \mathbb{C}_γ , where $w\gamma < \mu$ for some $w \in W$ or $\gamma = \mu$. Therefore, we conclude that

$$(5.8) \quad \text{ext}_{\mathfrak{B}}^\bullet(\mathbb{U}_{-\lambda}, D_\mu^*) = \{0\} \quad \lambda \not\leq \mu.$$

Assume that λ and μ are both anti-dominant. By Remark 3.4, we have $\mathbb{U}_{-\lambda} = \mathbb{W}_{-\lambda}$. Applying Lemma 5.10, we have

$$\begin{aligned}
 \text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{U}_{-\lambda}, D_{\mu}^*) &\cong \text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{W}_{-\lambda}, \mathcal{D}_{w_0}^{\dagger}(D_{w_0\mu}^*)) \\
 &\cong \text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{W}_{-w_0\lambda}, D_{w_0\mu}^*) \\
 (5.9) \qquad &= \{0\} \qquad \qquad \qquad \lambda \not\geq \mu.
 \end{aligned}$$

We calculate the ext-groups when $\lambda = \mu = 0$. We have $D_0 \cong \mathbb{C}_0$. Then, we can identify the projective resolution of \mathbb{C}_0 with the BGG resolution of D_0 in terms of the lowest weight Verma modules of $\widehat{\mathfrak{g}}$. In particular, the head of a projective resolution of D_0 in \mathfrak{B} has weight $-W^a\rho^a + \rho$, where ρ^a is an arbitrary weight in P^a so that $\langle \alpha_i^{\vee}, \rho^a \rangle = 1$ for every $i \in I^a$. In addition, each $w \in W^a$ corresponds to a single projective module in the BGG resolution. Therefore, the $\widehat{\mathfrak{h}}$ -eigen cyclic generators of weight 0 appears only once at the zero-th term. This implies

$$\text{ext}_{\mathfrak{B}}^k(\mathbb{U}_0, D_0^*) \cong \begin{cases} \mathbb{C} & (k = 0) \\ \{0\} & (\text{otherwise}) \end{cases}.$$

Summarizing the above, we have

$$\text{ext}_{\mathfrak{B}}^k(\mathbb{U}_{-\lambda}, D_0^*) = \begin{cases} \mathbb{C} & (k = 0, \lambda = 0) \\ \{0\} & (\text{otherwise}) \end{cases}.$$

We prove the main assertion by induction. Namely, we prove

$$\text{ext}_{\mathfrak{B}}^k(\mathbb{U}_{-\lambda}, D_{\gamma}^*) = \begin{cases} \mathbb{C} & (k = 0, \lambda = \gamma) \\ \{0\} & (\text{otherwise}) \end{cases}$$

for $\gamma \in P$ by assuming the same assertion for every $\mu \in \Lambda$ so that $u(\mu) < u(\gamma)$. The initial case $\gamma = \tau\Lambda_0$ for $\tau \in \Pi$ follows by the previous paragraph by applying a diagram automorphism of $\widehat{\mathfrak{g}}$ arising from τ (if $\tau \neq 1$). Hence, we can also assume $\gamma \notin \Pi\Lambda_0$ in addition.

We have some $i \in I^a$ so that $u(\mu) = s_i u(\gamma)$ and $u(\mu) < u(\gamma)$. Then, we have

$$\begin{aligned}
 \text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{U}_{-\lambda}, D_{\gamma}^*) &\cong \text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{U}_{-\lambda}, \mathcal{D}_i^{\dagger}(D_{\mu}^*)) \\
 &\cong \text{ext}_{\mathfrak{B}}^{\bullet}(\mathcal{D}_i(\mathbb{U}_{-\lambda}), D_{\mu}^*).
 \end{aligned}$$

In view of Proposition 5.9, we have $\mathcal{D}_i(\mathbb{U}_{-\lambda}) \cong \{0\}$ if $\langle \alpha_i^{\vee}, \lambda + \Lambda_0 \rangle > 0$, and $\mathcal{D}_i(\mathbb{U}_{-\lambda}) \cong \mathbb{U}_{-\lambda}$ if $\langle \alpha_i^{\vee}, \lambda + \Lambda_0 \rangle = 0$. In these cases, we have $\lambda \neq \gamma$, and the induction hypothesis yields

$$\text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{U}_{-\lambda}, D_{\gamma}^*) = \{0\}.$$

In case $\langle \alpha_i^{\vee}, \lambda + \Lambda_0 \rangle < 0$, then we have (a part of) the long exact sequence

$$\rightarrow \text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{U}_{-s_i\lambda}, D_{\mu}^*) \rightarrow \text{ext}_{\mathfrak{B}}^{\bullet}(\mathcal{D}_i(\mathbb{U}_{-\lambda}), D_{\mu}^*) \rightarrow \text{ext}_{\mathfrak{B}}^{\bullet}(\mathbb{U}_{-\lambda}, D_{\mu}^*) \rightarrow \text{ext}_{\mathfrak{B}}^{\bullet+1}(\mathbb{U}_{-s_i\lambda}, D_{\mu}^*).$$

As a consequence, we have non-zero result if and only if $s_i\lambda = \mu$ or $\lambda = \mu$ by the induction hypothesis. The latter case is prohibited by the comparison of $\langle \alpha_i^\vee, \lambda + \Lambda_0 \rangle < 0$ and $s_i u(\gamma) < u(\gamma)$. Therefore, we conclude

$$\mathrm{ext}_{\mathfrak{B}}^\bullet(\mathbb{U}_{-\lambda}, D_\gamma^*) \cong \mathrm{ext}_{\mathfrak{B}}^\bullet(\mathbb{U}_{-s_i\lambda}, D_{s_i\gamma}^*) = \mathrm{ext}_{\mathfrak{B}}^\bullet(\mathbb{U}_{-s_i\lambda}, D_\mu^*).$$

Therefore, our induction hypothesis proceeds the induction as required. \square

APPENDIX A. NUMERICAL EQUALITY

We discuss the equality of Theorem 5.12 on the level of characters. Consider the Cherednik kernel:

$$\kappa(x, q, t) = \frac{\prod_{\alpha \in \Delta_+^a} (1 - e^\alpha)}{\prod_{\alpha \in \Delta_+^a} (1 - te^\alpha)} \in \mathbb{C}[P](q, t) = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}](q, t),$$

through the identifications $e^{\omega_i} = x_i$ for $1 \leq i \leq n$ and $q = e^\delta$.

We consider the Euler-Poincaré pairing

$$(A.1) \quad [\mathfrak{B}'] \times [\mathfrak{B}_0] \ni (M, N) \mapsto (M, N)_{EP} := \sum_{i=0}^{\infty} (-1)^i \mathrm{gdim} \mathrm{ext}^i(M, N^*)^*,$$

as the formal sum. This pairing lands in $\mathbb{C}((q))$.

The Euler-Poincaré pairing satisfies the following properties:

- (i) It is q -linear;
- (ii) For a short exact sequence:

$$0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0,$$

we have

$$(M, N)_{EP} = (M_1, N)_{EP} + (M_2, N)_{EP}$$

and the same equality holds for a short exact sequence in the second argument. Thus the Euler-Poincaré pairing depends only on the characters of M and N ;

- (iii) We have the following equality:

$$(P_\Lambda, \mathbb{C}_\Gamma)_{EP} = \delta_{\Lambda, -\Gamma} \quad \Lambda, \Gamma \in P^a;$$

- (iv) If both M and N belong to \mathfrak{B}_0 , then we have

$$(M, N)_{EP} = (N, M)_{EP}.$$

The proofs of these properties are standard and is omitted (the last item requires [G, §2.1–2.2] as in the previous section).

The properties (i), (ii), (iii) completely characterizes the Euler-Poincaré pairing. Now consider the specialization of the Cherednik inner product on $\mathbb{C}[x^{\pm 1}](q)$:

$$(A.2) \quad (P(x, q), Q(x, q))_C := (P(x, q)Q(X, q)\kappa(x, q, 0))_0,$$

where the lower index 0 denotes the constant term with respect to q in the power series expansion of h . Applying (i), (ii), (iii) repeatedly, we obtain:

$$(A.3) \quad (M, N)_{EP} = (\text{ch } M, \text{ch } N)_C.$$

Theorem A.1. *For each $\lambda, \mu \in P$ so that $\lambda \neq \mu$, we have*

$$(A.4) \quad (\text{ch } U_\mu, \text{ch } D_\lambda)_C = 0 = (U_\mu, D_\lambda)_{EP}.$$

Proof. For $f(x, q, t) \in \mathbb{C}[P](q, t)$, we set

$$\overline{f(x, q, t)} = f(x^{-1}, q^{-1}, t^{-1}), \quad f^*(x, q, t) = f(x^{-1}, q^{-1}, t).$$

By the definition of the nonsymmetric Macdonald polynomials (see e.g. [Ch1]), we have

$$\left(E_\lambda(x, q, t) \overline{E_\mu(x, q, t)} \kappa(x, q, t) \right)_0 = 0$$

for $\lambda \neq \mu$. In other words:

$$(E_\lambda(x, q, t) E_\mu^*(x, q, t^{-1}) \kappa(x, q, t))_0 = 0.$$

Substituting $t = 0$, we obtain

$$(E_\lambda(x, q, 0) (w_0 E_{w_0(\mu)}(x, q^{-1}, \infty)) \kappa(x, q, 0))_0 = 0.$$

In view of Corollary 3.19 and Theorem 5.1, we conclude that the first equality. The second equality follows from (A.3) and (iv). \square

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